Complex Algebraic Surfaces, Homework Assignment 2, Spring 2009

1. Let S be a smooth connected complex projective surface, and $\epsilon : \hat{S} \to S$ the blow-up of a point $P \in S$. Let D be a divisor in $Div(S)$. Show that the pull-back homomorphism $\epsilon^* : Div(S) \to \mathrm{Div}(\hat{S})$ restricts to an isomorphism from $|D|$ onto $|\epsilon^* D|$. In particular, the linear system $\epsilon^*|D|$ is complete.

2. Let C and D be two linearly equivalent effective (and non-zero) divisors on a smooth connected complex projective surface S. Assume that $C \cap D$ consists of n points $\{Q_i \ : \ 1 \leq i \leq n\}$, $n \geq 0$, at each of which the two curves intersect transversally. Let $P \subset |D|$ be the pencil containing C and D. Let $f : S' \to S$ be the surface obtained by blowing-up each of the n points of $C \cap D$. Denote by $E_i \subset S'$ the exceptional divisor over Q_i. Show that the fixed part of $f^*(P)$ is precisely $F := \sum_{i=1}^{n} E_i$ and that $P' := f^*(P) - F$ is base point free. Show that the associated morphism $\phi : S' \to (P')^*$ restricts to each of the exceptional divisors E_i as an isomorphism from E_i onto $(P')^*$.

3. (This is another take on Example II.14 (2) in Beauville's text). Let Q_1, Q_2 be two distinct points in $\mathbb{P}^2$ and C the line in $\mathbb{P}^2$ through Q_1 and Q_2. Let $P_i \subset |\mathcal{O}_{\mathbb{P}^2}(1)|$ be the pencil of lines through Q_i, $i = 1, 2$. Let S be the blow-up of $\mathbb{P}^2$ at Q_1 and Q_2. Construct a morphism $\phi : S \to P_1^* \times P_2^*$. Show that ϕ is the blow-up of $P_1^* \times P_2^*$ at a point and that the strict transform of C is the exceptional divisor of ϕ.

4. Exercise II.20.1 page 23 in Beauville's text. (Read Remark I.16 (i) page 8).

5. Let C and D be two distinct irreducible curves on a smooth connected complex projective surface S and x a point in $C \cap D$. Let $m_x(C)$ be the multiplicity of C at x.

   (a) Let $\epsilon : \hat{S} \to S$ be the blow-up of S at x, $E \subset \hat{S}$ the exceptional divisor, and $\hat{C}$ the strict transform of C. Show that $m_x(C) = \hat{C} \cdot E$.

   (b) Keep the notation of part 5a. Prove the equality

   $$m_x(C \cap D) \quad = \quad m_x(C)m_x(D) + \sum_{t \in (E \cap C \cap D)} m_t(\hat{C} \cap \hat{D}) \qquad (1)$$

   Conclude the inequality $m_x(C \cap D) \geq m_x(C)m_x(D)$. The latter is Axioms (5) of intersection multiplicities in Fulton's *Algebraic Curves*, Chapter 3 section 3.

   (c) (Separating all points of intersection). Show that after a finite sequence of blow-ups, centered at points of $C \cap D$, and also at the points of intersections of the strict transforms of C and D, we arrive at a surface S', and a morphism $\pi : S' \to S$, satisfying: a) π is an isomorphism over $S \setminus [C \cap D]$. b) The strict transforms C' of C and D' of D in S' are disjoint. Hint: Note that Equation 1 implies the inequality $m_x(C \cap D) > \sum_{t \in (E \cap C \cap D)} m_t(\hat{C} \cap \hat{D})$.

6. Exercise II.20.2 page 23 in Beauville's text. This is a continuation of problem 5.