Complex Algebraic Surfaces, Homework Assignment 1, Spring 2009

The field k below is assumed algebraically closed.

1. Sketch the following curves in the affine plane $\mathbb{A}^2$:
   $A := V(y - x^2)$, $B := V(y^2 - x^3 + x)$, $C := V(y^2 - x^3)$, $D := V(y^2 - x^3 - x^2)$.
   Let $P := (0, 0)$. Compute the six intersection numbers at P, of the six pairs of curves, and compare each to the product of the two multiplicities of the curves at P.

2. Prove the bilinearity property (6) of the local intersection multiplicity $I(P, C \cap D)$ for two affine plane curves (not necessarily irreducible, nor reduced, but rather subschemes of pure dimension 1). In other words, let $F, G_1, G_2 \in k[x, y]$ be polynomials of positive degree. Set $G := G_1G_2$. Assume that the algebraic subsets $V(F)$ and $V(G)$ do not have any common irreducible component. Prove the equality
   $$I(P, F \cap G) \quad = \quad I(P, F \cap G_1) + I(P, F \cap G_2).$$
   Hint: Let $\mathcal{O}_P$ be the local ring of $\mathbb{A}^2$ at P. Prove that the sequence
   $$0 \to \mathcal{O}_P/(F, G_2) \stackrel{\psi}{\longrightarrow} \mathcal{O}_P/(F, G) \longrightarrow \mathcal{O}_P/(F, G_1) \to 0$$
   is short exact, where $\psi(z + (F, G_2)) := G_1 z + (F, G_1G_2)$.
   Remark: Observe that your argument goes through for pure one dimensional subschemes C and D over any smooth quasi-projective surface. (Beauville's definition of the local intersection multiplicity considers only reduced curves, but in the quation above we allow G_1 and G_2 to have commpon irreducible components, with arbitrary multiplicities, and F may have irreducible components with arbitrary multiplicities).

3. Let X be a smooth surface, C, D curves on X, which do not have common irreducible components, and $P \in C$ a smooth point. Let $\mathcal{O}_{X,P}$ be the local ring of X at P, $\mathcal{O}_{C,P}$ the local ring of C at P, $f \in \mathcal{O}_{X,P}$ a local equation of D, and $\bar{f}$ its image in $\mathcal{O}_{C,P}$. Use the fact that localization is an exact functor to prove the following equality:
   $$m_P(C \cap D) \quad = \quad \mathrm{ord}_p(\bar{f}).$$
   Conclude that when C is smooth and irreducible, the restriction of the invertible sheaf $\mathcal{O}_X(D)$ to C is isomorphic to $\mathcal{O}_C\left(\sum_{P \in C} m_P(C \cap D)P\right)$. (Compare, but do not use, Beauville, Lemma I.6).

4. (Hartshorne, Proposition 6.5. The proof is easy, so try to do it yourself and then check your answer). Let X be a smooth quasi-projective variety, and $Z \subset X$ a closed algebraic subset. Set $U := X \setminus Z$. Prove the following statements.

   (a) The homomorphism $\rho : \mathrm{Pic}(X) \to \mathrm{Pic}(U)$, defined by
   $\sum n_i Y_i \mapsto \sum_i n_i(Y_i \cap U)$, is surjective. The divisor $Y_i \cap U$ above is the zero divisor of U, if the intersection is empty.

(b) If the codimension of Z in X is ≥ 2, then ρ is an isomorphism.

(c) If Z is an irreducible subset of codimension 1, then the sequence

$$\mathbb{Z} \to \mathrm{Pic}(X) \xrightarrow{\rho} \mathrm{Pic}(U) \to 0,$$

is exact, where the left homomorphism sends 1 to Z.

5. Let X be a quasi-projective variety and $n \geq 1$. Show that $\mathrm{Pic}(X \times \mathbb{P}^n)$ is isomorphic to $\mathrm{Pic}(X) \times \mathbb{Z}$.
Hints: i) Recall Hartshorne, Proposition II.6.6: Let $X \subset \mathbb{P}^n$ be a quasi-projective variety. Then the homomorphism

$$\mathrm{Pic}(X) \to \mathrm{Pic}(X \times \mathbb{A}^1),$$

sending $\sum n_i Y_i$ to $\sum n_i(Y_i \times \mathbb{A}^1)$, is an isomorphism.
ii) Let $H \subset \mathbb{P}^n$ be a hyperplane, set $U := \mathbb{P}^n \setminus H$, and prove that the following sequence is short exact.

$$0 \to \mathbb{Z} \to \mathrm{Pic}(X \times \mathbb{P}^n) \xrightarrow{\rho} \mathrm{Pic}(X \times U) \to 0.$$

6. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Prove that $\mathrm{Pic}(Q)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.
Hint: Use question 5.

7. Let C be a smooth projective curve of genus g. Set $X := C \times C$, and let $\Delta \subset X$ be the (reduced) diagonal curve $\{(P, P) : P \in C\}$. Show that the class of Δ in $\mathrm{Pic}(X)$ belongs to the image of the homomorphism

$$\begin{array}{rcl} \mathrm{Pic}(C) \times \mathrm{Pic}(C) & \longrightarrow & \mathrm{Pic}(X) \\ (D_1, D_2) & \mapsto & (D_1 \times C) + (C \times D_2), \end{array}$$

if and only if $g = 0$.
Hint: Assume that $\Delta \sim (D_1 \times C) + (C \times D_2)$, $D_i \in \mathrm{Pic}(C)$. Use Theorem I.4 in Beauville's text to show that $\deg(D_i) = 1$. Then show that there exist two distinct points P, $Q \in C$, such that the divisors P and Q in $Div(C)$ are both linearly equivalent to D_1.

8. Let C be a smooth projective curve of genus one. Fix a point $P_0 \in C$. Consider the map $a : C \to \mathrm{Pic}(C)$, sending $P \in C$ to $P - P_0$. Let $\deg : \mathrm{Pic}(C) \to \mathbb{Z}$ be the degree map, sending $\sum_{P \in C} n_P \cdot P$ to $\sum_{P \in C} n_P$. When the genus of C is 1, we have shown in class, using the Riemann-Roch Theorem, that the image of a is equal to the kernel of deg. In particular, the choice of the point P_0 endows C with a group structure. Recall also that the genus of a smooth curve of degree d in $\mathbb{P}^2$ is $(d-1)(d-2)/2$.

   Let C be $V(zy^2 - x(x-z)(x-\lambda z)) \subset \mathbb{P}^2$, where $\lambda \in \mathbb{C} \setminus \{0, 1\}$. Set $P_0 := (0, 1, 0)$. Given a curve D in $\mathbb{P}^2$, which does not contain C, denote by $C \cap D$ the divisor $\sum_{P \in C} m_P(C \cap D)P$ in $Div(C)$.

(a) Set $H := V(z)$. Show that $m_{P_0}(H \cap C) = 3$. Conclude that the divisor class of $3P_0$ generates the image of the restriction homomorphism $\mathrm{Pic}(\mathbb{P}^2) \to \mathrm{Pic}(C)$.

(b) Let P, Q, and R be points of C (not necessarily distinct). Show that $a(P) + a(Q) + a(R) = 0$ in $\mathrm{Pic}(C)$, if and only if there exists a line L in $\mathbb{P}^2$, such that $(L \cap C) = P + Q + R$. The points P, Q, and R are said in this case to be *co-linear*.

(c) Let $L_{P,Q}$ be the unique line in $\mathbb{P}^2$, such that $L_{P,Q} \cap C = P + Q + R$, for some point $R \in C$. (If $P = Q$, then $L_{P,Q}$ is the tangent line to C at P). Show that $a(P) + a(Q) = a(S)$, if and only if P_0, S, and R are co-linear, where $L_{P,Q} \cap C = P + Q + R$. Drow a picture.

(d) Give a geometric interpretation for the inversion $P \mapsto a^{-1}(-a(P))$.