AN APPENDIX TO: THE BEAUVILLE-BOGOMOLOV CLASS AS A CHARACTERISTIC CLASS

EYAL MARKMAN

Contents

1. Introduction

Let X be any compact Kähler manifold deformation equivalent to the Hilbert scheme $S^{[n]}$ of length n subschemes on a $K3$ surface S, $n \geq 2$. Such a manifold will be called below of $K3^{[n]}$*-type*. Let $\Delta \subset X \times X$ be the diagonal. The paper "The Beauville-Bogomolov class as a characteristic class" carries out a construction of a $\mathbb{P}^{2n-3}$-bundle $\mathbb{P}V$ over $\widetilde{\Sigma} := [X \times X] \setminus \Delta$, which corresponds to a slope-stable twisted reflexive sheaf over $X \times X$, with monodromy-invariant characteristic classes (see [M]). We first constructed $\mathbb{P}V$ when X is a moduli space of sheaves. We then used Verbitsky's theory of hyperholomorphic sheaves in order to deform the construction of $\mathbb{P}V$ to every X as above.

In this appendix we provide another geometric interpretation of the above construction. Let Σ be the complement of the diagonal in the second symmetric product of X. Then $\widetilde{\Sigma}$ is the universal cover of Σ. Now Σ is expected to be a stratum of the singular locus of a $\mathbb{Q}$-factorial compact holomorphic symplectic variety Y, at least when X is a sufficiently "small" deformation of $S^{[n]}$ (Conjecture 2.2). We reconstruct the projective bundle $\mathbb{P}V$ over $\widetilde{\Sigma}$ from the geometry of the projectivized normal cone of Σ in Y. Conjecture 2.2, if true, would thus provide an alternative construction of the pair $(X, \mathbb{P}V)$ constructed in [M], for X in the local Kuranishi deformation space of $S^{[n]}$.

2. Singular moduli spaces and their deformations

Let S be a projective $K3$ surface with a cyclic Picard group generated by an ample line-bundle H of degree 2. Let $K_{top}S$ be the topological K group of S and denote by $v \in K_{top}S$ the class of the ideal sheaf of a length n subscheme of S, so that the moduli space $\mathcal{M}_H(v)$, of H stable coherent sheaves of class v, is simply the Hilbert scheme $S^{[n]}$. Assume that $n \geq 2$. The equivalence class (also known as the S-equivalence class), of an H-semistable sheaf, is the isomorphism class of the associated graded sheaf, with respect to the Harder-Narasimhan filtration. Let $\mathcal{M}_H(2v)$ be the moduli space of equivalence classes of H-semistable sheaves over S, with class $2v$. If $n \geq 3$, then $\mathcal{M}_H(2v)$ is an irreducible locally factorial singular projective symplectic variety with terminal singularities, which does not admit a crepant resolution. [KLS]. When $n = 2$ the moduli space $\mathcal{M}_H(2v)$ is $\mathbb{Q}$-factorial and it does admit a crepant resolution [O'G, LS]. The singularities of $\mathcal{M}_H(2v)$ determine a stratification

$$\mathcal{M}_H(2v) \ \supset \ \mathcal{M}(2v)_{sing} \ \supset \ \mathcal{M}(v),$$

$\mathcal{M}(2v)_{sing}$ is isomorphic to the second symmetric product $\mathrm{Sym}^2\mathcal{M}(v)$, and $\mathcal{M}(v) \hookrightarrow \mathrm{Sym}^2\mathcal{M}(v)$ is the diagonal embedding. A point in $\mathcal{M}(2v)_{sing}$ corresponds to the S-equivalence class of the direct sum $I_{Z_1} \oplus I_{Z_2}$ of two ideal sheaves, with Z_j, $j = 1, 2$, a length n subscheme of S.

Let $\mathcal{Y} \rightarrow Def(\mathcal{M}_H(2v))$ be the semi-universal family over the local Kuranishi deformation space of $\mathcal{M}_H(2v)$. Namikawa studied the deformation theory of $\mathbb{Q}$-factorial projective symplectic varieties with terminal singularities [Nam1, Nam2, Nam3]. His results imply that $Def(\mathcal{M}_H(2v))$ is smooth ([Nam1], Theorem 2.5). Furthermore, the semi-universal family $\mathcal{Y}$ is locally trivial [Nam3]. So deformations of $\mathcal{M}_H(2v)$ remain singular and the deformation $p : \mathcal{Y}_{sing} \rightarrow Def(\mathcal{M}_H(2v))$ of their singular loci is locally trivial over $Def(\mathcal{M}_H(2v))$, for $n \geq 3$. Local triviality means that given a point $y \in \mathcal{Y}_{sing}$, there exists an analytic open neighborhoods U of y in $\mathcal{Y}_{sing}$, U_1 of $p(y)$ in $Def(\mathcal{M}_H(2v))$, and U_2 of y in the fiber Y_{sing} over $p(y)$, and an isomorphism $U \cong U_1 \times U_2$, which conjugates p to the projection onto U_1.

Corollary 2.1. *The fiber Y, over a generic point of $Def(\mathcal{M}_H(2v))$, is singular, with a stratification*

$$Y \ \supset \ Y_{sing} \ \supset \ X, \tag{2.1}$$

where the reduced singular locus Y_{sing} is isomorphic to $\mathrm{Sym}^2(X)$, X is smooth of $K3^{[n]}$-type, and the inclusion $X \subset Y_{sing}$ is the diagonal embedding.

Proof. Simply use Namikawa's local triviality twice. Once for the semi-universal family $\mathcal{Y} \rightarrow Def(\mathcal{M}_H(2v))$, in order to conclude the flatness of $p : \mathcal{Y}_{sing} \rightarrow Def(\mathcal{M}_H(2v))$, and once to conclude the local trivialiity of p. □

Corollary 2.1 gives rise to a natural morphism of local deformation spaces

$$Def(\mathcal{M}_H(2v)) \quad \longrightarrow \quad Def(\mathcal{M}_H(v)), \tag{2.2}$$

sending Y to the smallest stratum of its singular locus. When $n \geq 3$, both moduli spaces are smooth and 23-dimensional. Recall that $\mathcal{M}_H(v) = S^{[n]}$.

Conjecture 2.2. *The morphism (2.2) is surjective, for a generic polarized $K3$ surface (S, H).*

Note that it would suffice to prove that the differential of the morphism (2.2) is invertible, a calculation we have not carried out.

3. O'Grady's results on the structure of the normal cone

Set

$$\begin{aligned} \Sigma &:= \mathcal{M}(2v)_{sing} \setminus \mathcal{M}(v), \\ \widetilde{\Sigma} &:= [\mathcal{M}(v) \times \mathcal{M}(v)] \setminus \Delta_{\mathcal{M}(v)}. \end{aligned}$$

Then $\widetilde{\Sigma} \to \Sigma$ is the universal cover and we let $\tau : \widetilde{\Sigma} \to \widetilde{\Sigma}$ denotes its Galois involution. Let $\mathcal{E}$ be the universal ideal sheaf over $S \times \mathcal{M}_H(v)$. Denote by π_{ij} the projection from $\mathcal{M}_H(v) \times S \times \mathcal{M}_H(v)$ onto the product of the i-th and j-th factors. Denote by V the restriction to $\widetilde{\Sigma}$ of the sheaf

$$\mathcal{E}xt^1_{\pi_{13}}\left(\pi_{12}^*\mathcal{E}, \pi_{23}^*\mathcal{E}\right). \tag{3.1}$$

Then V is a locally free sheaf of rank $2n-2$. Let $\tilde{q} \in \mathrm{Sym}^2[V \oplus V^*]^*$ be the symmetric bilinear form $\tilde{q}(x, y) = y(x)$ and $C_{\widetilde{\Sigma}} \subset \mathbb{P}V \times_{\widetilde{\Sigma}} \mathbb{P}V^*$ the subscheme defined by $\tilde{q} = 0$. A fiber of $C_{\widetilde{\Sigma}}$, over $\sigma \in \widetilde{\Sigma}$, is the incidence divisor

$$\mathcal{Q} \quad \subset \quad [\mathbb{P}^{2n-3} \times (\mathbb{P}^{2n-3})^*]. \tag{3.2}$$

When $n = 2$, $\mathbb{P}V$ is a $\mathbb{P}^1$-bundle, hence self-dual, and $C_{\widetilde{\Sigma}}$ is the graph of the isomorphism $\mathbb{P}V \cong \mathbb{P}V^*$.

The pullback τ^*V is isomorphic to V^*. Thus, the vector bundle $V \oplus V^*$, the quadratic form $\tilde{q}$, the fiber product $\mathbb{P}V \times_{\widetilde{\Sigma}} \mathbb{P}V^*$, and its subvariety $C_{\widetilde{\Sigma}}$, descend to a vector bundle over Σ with a quadratic form q, a $\mathbb{P}^{2n-3} \times (\mathbb{P}^{2n-3})^*]$-bundle $\mathcal{P}$ over Σ, and a subvariety

$$C_\Sigma \quad \subset \quad \mathcal{P}.$$

Proposition 3.1. *(*[O'G]*, Proposition 1.4.1 and Theorem 1.2.1) C_Σ is isomorphic to the projectivized normal cone of Σ in $\mathcal{M}_H(2v)$.*

4. Extracting the data $\{\mathbb{P}V, \mathbb{P}V^*\}$ from the normal cone

Assume that conjecture 2.2 holds. Let X be an irreducible holomorphic symplectic manifold, parametrized by a point $[X]$ in $Def(S^{[n]})$ in the image of a point $[Y]$ in $Def(\mathcal{M}_H(2v))$ via the morphism (2.2). Let $\Sigma := [\mathrm{Sym}^2 X] \setminus \Delta$ and $\widetilde{\Sigma} := [X \times X] \setminus \Delta$ be the complements of the diagonals. Then Σ is a stratum in Y_{sing} and the projectivized normal cone C_Σ, of Σ in Y, is a $\mathcal{Q}$-bundle, where $\mathcal{Q}$ is the incidence divisor (3.2), by Corollary 2.1.

If $n = 2$, the pullback of C_Σ to $\widetilde{\Sigma}$ is the $\mathbb{P}^1$-bundle we are looking for[1]. Assume $n \geq 3$. Then the relative Picard sheaf $\mathrm{Pic}(C_\Sigma/\Sigma)$ is a $\mathbb{Z} \oplus \mathbb{Z}$ local

[1] In the case $n = 2$ one need not consider the whole of $Def(\mathcal{M}_H(2v))$, but rather the divisor along which the fiber Y of the semi-universal family remains singular.

system, and it has a canonical double section, whose value at a point $\sigma \in \Sigma$ is the pair of two generators of the effective cone of the fiber $\mathcal{Q}$ of C_Σ over σ. The double section is connected, as it is connected in the case $\Sigma = \mathcal{M}(2v)_{sing} \setminus \mathcal{M}(v)$. Hence, the double section is isomorphic to the universal cover $\widetilde{\Sigma}$ of Σ. Let $C_{\widetilde{\Sigma}}$ be the fiber product $C_\Sigma \times_\Sigma \widetilde{\Sigma}$. Then $\mathrm{Pic}(C_{\widetilde{\Sigma}}/\widetilde{\Sigma})$ has an unordered pair of two sections $\{L_1, L_2\}$ (labeled by a choice of identification of $\widetilde{\Sigma}$ with the double section), such that L_i "restricts" to the fiber $\mathcal{Q}$ of $C_{\widetilde{\Sigma}} \rightarrow \widetilde{\Sigma}$ as the line bundle $\mathcal{O}_{\mathcal{Q}}(1,0)$ or $\mathcal{O}_{\mathcal{Q}}(0,1)$ of the incidence divisor (3.2). Each L_i determines a $\mathbb{P}^{2n-3}$-bundle $\mathbb{P}_i$ over $\widetilde{\Sigma}$ (of linear systems along the fibers), and a morphism $\eta_i : C_{\widetilde{\Sigma}} \rightarrow \mathbb{P}_i^*$. The morphisms η_1, η_2 are the two rulings of $C_{\widetilde{\Sigma}}$. The embedding

$$(\eta_1, \eta_2) \ : \ C_{\widetilde{\Sigma}} \quad \longrightarrow \quad \mathbb{P}_1^* \times \mathbb{P}_2^*$$

determines an isomorphism $\mathbb{P}_1 \cong \mathbb{P}_2^*$. When $X = \mathcal{M}(v) \cong S^{[n]}$ and $Y = \mathcal{M}_H(2v)$, the dual pair $\{\mathbb{P}_1, \mathbb{P}_2\}$ is precisely the pair $\{\mathbb{P}V, \mathbb{P}V^*\}$, where V is given in (3.1).

References

[KLS] Kaledin, D., Lehn, M., Sorger, C.: *Singular symplectic moduli spaces.* Invent. Math. 164 (2006), no. 3, 591–614.

[LS] Lehn, M., Sorger, C.: *La singularité de O'Grady.* J. Algebraic Geom. 15 (2006), no. 4, 753-770.

[M] Markman, E.: *The Beauville-Bogomolov class as a characteristic class.* Preprint, May 2010, http://www.math.umass.edu/∼markman/

[Nam1] Namikawa, Y.: *Deformation theory of singular symplectic n-folds.* Math. Ann. 319 (2001), no. 3, 597–623.

[Nam2] Namikawa, Y.: *Extension of 2-forms and symplectic varieties.* J. reine angew. Math. 539 (2001), 123-147

[Nam3] Namikawa, Y.: *On deformations of Q-factorial symplectic varieties.* J. Reine Angew. Math. 599 (2006), 97–110.

[O'G] O'Grady, K.: *Desingularized moduli spaces of sheaves on a K3.* J. Reine Angew. Math. 512 (1999), 49–117.