Algebraic Geometry Homework Assignment 9, Fall 2007
Due Tuesday, December 4.

The field k below is assumed algebraically closed.

1. Let X be an affine non-singular (irreducible) curve[1] with coordinate ring A. Let $I \subset A$ be a non-zero ideal. Then there exists a finite set of distinct points $P_1, \ldots, P_n$, and positive integers d_i, $1 \leq i \leq n$, such that $I = M_{P_1}^{d_1} \cap \cdots \cap M_{P_n}^{d_n}$, where M_{P_i} is the maximal ideal of P_i, by the Primary Decomposition Theorem (Atiyah-Macdonald, Theorem 7.13).

   (a) Prove that there exist functions $g_i \in \bigcap_{\substack{j=1 \\ j\neq i}}^n M_{P_j}^{d_j}$, such that $\sum_{i=1}^n g_i = 1$. Hint: Note that $\bigcap_{i=1}^n V\left(\bigcap_{\substack{j=1 \\ j\neq i}}^n M_{P_j}\right) = \emptyset$.

   (b) Prove that the natural homomorphism $A/I = A/\bigcap_{i=1}^n M_{P_i}^{d_i} \rightarrow \prod_{i=1}^n A/M_{P_i}^{d_i}$ is an isomorphism.

   (c) Prove that $\dim_k\left(A/M_{P_i}^{d_i}\right) = d_i$. Hint: Use the exactness property of locallization[2] to prove that $\dim_k\left(M_{P_i}^t/M_{P_i}^{t+1}\right) = 1$.

   (d) Set $\mathrm{ord}_P(I) := \min\{\mathrm{ord}_P(f) \ : \ f \in I\}$. Prove that $\dim_k(A/I) = \sum_{P\in X} \mathrm{ord}_P(I)$.

   (e) Let $S \subset A$ be a multiplicative system. Prove that $S^{-1}A/S^{-1}I$ is isomorphic to $\prod_{\{i \ : \ S\cap M_{P_i}=\emptyset\}} A/M_{P_i}^{d_i}$. Hint: Show that the image of $a \in A$ in $A/M_{P_i}^{d_i}$ is invertible, if $a \notin M_{P_i}$, and nilpotent if $a \in M_{P_i}$. Next use the exactness property of locallization.

2. (The degree of a morphism of curves and the length of a fiber) Let $f : X \rightarrow Y$ be a dominant morphism of varieties. We identify $K(Y)$ as a subfield of $K(X)$ via the natural homomorphism $f^* : K(Y) \rightarrow K(X)$ induced by f. The *degree* of f is defined to be the degree of the field extension $[K(X) : K(Y)]$. When X and Y are non-singular projective curves (one-dimensional varieties over k), you will show below that the degree is equal to the number of points in each fiber, counted appropriately. Now both invariants are local in Y, so the discussion reduces to the following setup (see part 2a for the reduction). Assume that X and Y are affine, non-singular, and $f : X \rightarrow Y$ is a finite morphism. Let A and B be the coordinate rings of X and Y respectively. Note that A is integrally closed in $K(X)$, since X is non-singular, and A is integral over B, since f is finite (see Mumford, section I.7 Definition 2). Thus A is the integral closure of B in $K(X)$. Note also that A is a finitely generated B-module, by Hartshorne, Theorem I.3.9A.

   (a) Let $\bar{f} : C_{K(X)} \rightarrow C_{K(Y)}$ be the morphism extending f to the projective non-singular curves defined in section I.6 of Hartshorne. Identify Y with its image in $C_{K(Y)}$ via the natural embedding. Prove that $\bar{f}^{-1}(Y)$ is isomorphic to X. Conclude that $f^{-1}(f(P))$ and $\bar{f}^{-1}(f(P))$ are equal subsets of $C_{K(X)}$. Hint: Given a DVR $R \in C_{K(X)}$, show that $R' := R \cap K(Y)$ is a DVR in $C_{K(Y)}$, by showing that

[1] Parts 1a, 1b, and 1e generalize, *using the same argument*, to the case where X is any affine variety and $V(I)$ is zero dimensional. The Primery Decomposition Theorem then implies, that I is the intersection $I_1 \cap \cdots \cap I_n$ of ideals, whose radical $\sqrt{I_j}$ is a maximal ideal M_{P_j}.

[2] Let A be a ring, $S \subset A$ a multiplicative system, and $0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$ an exact sequence of A-modules. Then the sequence $0 \rightarrow S^{-1}M_1 \rightarrow S^{-1}M_2 \rightarrow S^{-1}M_3 \rightarrow 0$ of $S^{-1}A$-modules is exact as well. You will need to use it only when M_1 and M_2 are ideals of A, so that $S^{-1}M_1$ is the ideal generated by the image of M_1 in $S^{-1}A$, and $S^{-1}M_3 := \overline{S}^{-1}M_3$, where $\overline{S}$ is the image of S in A/M_1.

$m' := m_R \cap K(Y)$ is a maximal ideal of R', $R' \setminus m'$ consists of invertible elements of R', and R' is integrally closed in $K(Y)$. Use Lemma I.6.4 in Hartshorne to prove that the map $\bar{f}$ sends a DVR $R \in C_{K(X)}$ to the DVR $R \cap K(Y)$ in $C_{K(Y)}$. Recall next that we proved in class the following generalization of Hartshorne, Lemma I.6.5: Let $S \subset K(X) \setminus k$ be a finite non-empty subset. Then $\{R \in C_{K(X)} \ : \ S \subset R\}$ is an open affine subset of $C_{K(X)}$, whose coordinate ring is the integral closure of the k-subalgebra of $K(X)$ generated by S.

(b) Let $M_Q \subset B$ be the maximal ideal of a point $Q \in Y$ and consider $S := B \setminus M_Q$ as a multiplicative system in both B and A. By definition, $\mathcal{O}_Q := S^{-1}B$. Show that $S^{-1}A$ is a free $\mathcal{O}_Q$-module of finite rank. Hint: note that a DVR is also a PID and use your first-year algebra.

(c) Consider now $\Sigma := B \setminus \{0\}$ as a multiplicative system in both A and B. By definition, $K(Y) := \Sigma^{-1}B$. Prove the equality $\Sigma^{-1}A = K(X)$. Conclude that the rank of $S^{-1}A$ as an $\mathcal{O}_Q$-module (in part 2b) is equal to the degree $[K(X) : K(Y)]$ of f (Hint: see Homework 3 Question 6a).

(d) Prove that the natural homomorphism $A/(M_Q A) \to (S^{-1}A)/[m_Q(S^{-1}A)]$ is an isomorphism, where m_Q is the maximal ideal of $\mathcal{O}_Q$. Conclude that $\dim_k[A/(M_Q A)] = [K(X) : K(Y)]$. Hint: Recall the exactness property of localization. Note: The ring $A/(M_Q A)$ is the coordinate ring of the fiber $f^{-1}(Q)$ as a subscheme of X (to be defined in class shortly), and the *length* of the fiber is defined to be $\dim_k[A/(M_Q A)]$. You have thus proven that *the length of the fiber is equal to the degree of f.*

(e) **Definition:** Let P be a point in the fiber $f^{-1}(Q)$. The *multiplicity* $\mu_f(P)$ of P in the fiber of f over Q is $\mathrm{ord}_P(t_Q)$, where t_Q is any uniformizing parameter of $\mathcal{O}_Q$. If $\mu_f(P) > 1$, we say that P is a *ramification point.* Q is a *branch point*, if the fiber $f^{-1}(Q)$ contains a ramification point.

Conclude, using Questions 1d and 2d, the equality

$$\sum_{P \in f^{-1}(Q)} \mu_f(P) \quad = \quad [K(X) : K(Y)],$$

i.e., *the number of points in each fiber, counted with multiplicities, is equal to the degree of f.*

Note: Assume that the field extension $K(Y) \subset K(X)$ is separable[3] (automatic when $char(k) = 0$ or $char(k) = p$ and p does not divide $[K(X) : K(Y)]$). Then the number of ramification points is finite. If $char(k) = p$, assume further that p does not divide the multiplicity $\mu_f(P)$, of any ramification point $P \in X$. One of the many characterizations of the genus g_X of a non-singular projective curve X is given by the Riemann-Hurwitz formula: The morphism $f : X \to Y$ satisfies

$$(2g_X - 2) = \deg(f)(2g_Y - 2) + \sum_{P \in X} (\mu_f(P) - 1).$$

The *ramification index* $\mu_f(P) - 1$ vanishes, unless P is a ramification point, so the sum is the number of ramification points, counted with multiplicites. The genus of $\mathbb{P}^1$ is zero and the genus of X could be determined by counting the ramification points of a non-constant rational function $f : X \to \mathbb{P}^1$. In general, however, when $char(k)$ is a prime dividing $[K(X) : K(Y)]$, it is possible for the morphism f to be ramified at all points of X. See Proposition IV.2.5 in Hartshorne.

[3] An algebraic field extension $K \subset L$ is *separable*, if every element $\alpha \in L$ is the root of an irreducible polynomial $F(x) \in K[x]$, such that every root of F has multiplicity 1.

3. Let X be a non-singular projective curve and $f \in K(X) \setminus k$ a non-constant rational function. Prove the equality $\sum_{P \in X} \text{ord}_P(f) = 0$. Hint: Interpret the number of zeroes of f (respectively poles), counted with multiplicities, as the number of points in the fiber over 0 (respectively ∞), of the morphism $f : X \to \mathbb{P}^1$.

4. (*Intersection Multiplicities*, Hartshorne, Problem 5.4, modified[4]) Let $C = V(F), D = V(G) \subset \mathbb{A}^2$ be two distinct (irreducible) curves, where $F, G \in k[X, Y]$. Given a point $P \in C \cap D$, define the *intersection multiplicity* $(C \cdot D)_P$ to be $\dim_k \left(\mathcal{O}_{\mathbb{A}^2,P}/(F, G)\right)$, where $\mathcal{O}_{\mathbb{A}^2,P}$ is the local ring of P in $\mathbb{A}^2$.

   (a) Set $A := \Gamma(C) := k[X, Y]/(F)$ and let $\tilde{A}$ be the integral closure of A in its quotient field $K(C)$. Let $\widetilde{C}$ be the affine curve with coordinate ring $\tilde{A}$ and $\nu : \widetilde{C} \to C$ the morphism, such that $\nu^* : A \hookrightarrow \tilde{A}$ is the the inclusion. $\tilde{C}$ is called the *normalization* of C, or the resolution of singularities of C. Let $P_1, \ldots, P_n$ be the points of $\nu^{-1}(P)$ over $P \in C \cap D$ and let g be the restriction of G to C. Prove the equality $(C \cdot D)_P = \sum_{i=1}^n \text{ord}_{P_i}(\nu^* g)$.
   Hint: Let $M_P \subset A$ be the maximal ideal and $S := A \setminus M_P$. Regard S as a multiplicative system in both A and $\tilde{A}$. Show first that $(C \cdot D)_P = \dim_k[A_P/(g)]$, where $A_P := S^{-1}A$ is the local ring of C at P. Set $\tilde{A}_P := S^{-1}\tilde{A}$. Consider the commutative diagram
   $$\begin{array}{ccccccc} 0 \to & (g) & \to & A_P & \to & A_P/(g) & \to 0 \\ & \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & \\ 0 \to & (\nu^* g) & \to & \tilde{A}_P & \to & \tilde{A}_P/(\nu^* g) & \to 0. \end{array}$$
   Show that $\dim_k[coker(\alpha)] = \dim_k[coker(\beta)]$, and both are finite dimensional[5] (note that $\tilde{A}$ is a finitely generated A-module, by Hartshorne, Theorem I.3.9A). Conclude, using the Snake Lemma, that $\dim_k[A_P/(g)] = \dim_k[\tilde{A}_P/(\nu^* g)]$. Finally, prove the equality $\dim_k[\tilde{A}_P/(\nu^* g)] = \sum_{i=1}^n \text{ord}_{P_i}(\nu^* g)$, using Question 1.

   (b) Let $\mu_P(C)$ be the multiplicity of P on C in the sense of Homework 8 Problem 1. Show that $(C \cdot D)_P \geq \mu_P(C) \cdot \mu_P(D)$, with strict inequality when C and D have a common tangent direction at P. Hint: Use part 4a to reduce it to the case where D is a line. Then exploit the symmetry of $(C \cdot D)_P$.

   (c) If $P \in C$, show that for all but a finite number of lines L through P, $(L \cdot C)_P = \mu_P(C)$.

   (d) **Definition:** Given two curves C, D in $\mathbb{P}^2$, $C \neq D$, set $(C \cdot D) := \sum_{P \in C \cap D} (C \cdot D)_P$, where$(C \cdot D)_P$ is defined using a suitable affine cover[6] of $\mathbb{P}^2$.
   If C is a curve of degree d in $\mathbb{P}^2$, and if L is a line in $\mathbb{P}^2$, $L \neq C$, show that $(L \cdot C) = d$.

   (e) Show that an irreducible curve C of degree $d > 1$ in $\mathbb{P}^2$ can not have a point of multiplicity $\geq d$. When $d = 3$ and C is singular, conclude that it has precisely one double point[7].

[4]Part 4a generalizes, with the same argument, for C a curve in a smooth variety X and D a hypersurface in X not containing C. Parts 4d and 4f generalize, with the same argument, for C a curve in $\mathbb{A}^n$ or $\mathbb{P}^n$ and D a hypersurface not containing C. The whole problem is generalized in section I.7 of Hartshorne. Parts 4a and 4f rely on section I.6 of Hartshorne.

[5]Note that the integer $\delta_P := \dim_k[coker(\beta)]$ is a canonical invariant of the point P, which vanishes if and only if P is a non-singular point of C. If $\pi : C' \to C$ is the blow-up of C at P and $Q \in \pi^{-1}(P)$, it can be shown that $\delta_Q < \delta_P$. Consequently, the singularities of C can be resolved by a finite sequence of blow-ups. See Hartshorne, Exercise IV.1.8 and Proposition V.3.8.

[6]We can choose homogeneous coordinates x, y, z on $\mathbb{P}^2$, so that $C \cap D$ is contained in $\mathbb{P}^2 \setminus V(z) \cong \mathbb{A}^2$. Then the global intersection number (C, D) is the dimension of $k[x, y]/(F, G)$, by Problem 1. The latter is the coordinate ring of the zero-dimensional subscheme $C \cap D$, to be defined shortly in class.

[7]If $char(k) \neq 2$, it is not hard to further show that the singular point must be either an ordinary node or a cusp

(f) Let C and D be as in part 4d. Prove **Bézout's Theorem:**

$$(C \cdot D) \quad = \quad \deg(F) \cdot \deg(G).$$

Hint: Everything above goes through, if $D = V(G')$ is reducible, G' a homogeneous polynomial, such that C is not an irreducible component of D. Set $f := G/\left(\prod_{i=1}^{\deg(G)} L_i\right)$, for sufficiently general lines L_i, and use Question 3.

5. (a) (The quotient of a curve by a group of automorphisms) Let X be a non-singular projective curve over k and G a finite subgroup of automorphisms of X. Prove that there exists a unique non-singular projective curve Y with the following property. There exists a G-invariant morphism $\varphi : X \to Y$ of degree equal to the cardinality $|G|$ of G. We denote Y by X/G. Hint: Use Artin's Theorem[8] from Galois Theory.

   (b) Assume that $char(k) \neq 2$. Let $\varphi : X \to Y$ be a morphism of degree 2 between non-singular projective curves over k. Prove that there exists an automorphism $\iota : X \to X$ of order 2, such that $\varphi \circ \iota = \iota$ and Y is isomorphic to $X/\{1, \iota\}$.

   (c) Let G be a finite subgroup of $PGL(2, k)$. Show that $\mathbb{P}^1/G$ is isomorphic to $\mathbb{P}^1$.

   (d) Let $X \subset \mathbb{P}^2$ be the curve $x^d + y^d + z^d = 0$, where $d \geq 1$ and $char(k)$ is either 0 or does not divide d. Construct a group G of automorphisms of X, which is isomorphic to the semi-direct product of the symmetric group on three letters and $\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$. Show that X/G is isomorphic to $\mathbb{P}^1$. Hint: Consider first $X/[\mathbb{Z}/d\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}]$.

6. Assume $char(k) \neq 2$. A hyperelliptic curve X is a non-singular projective curve, which admits a morphism $\varphi : X \to \mathbb{P}^1$ of degree 2.

   (a) (Construction of hyperelliptic curves of genus g) Let x_0, x_1 be homogeneous coordinates on $\mathbb{P}^1$ and set $x := x_1/x_0$. Fix integers $g \geq 0$ and $\epsilon \in \{1, 2\}$. Let $f(x) = \prod_{i=1}^{2g+\epsilon}(x - \lambda_i) \in k[x]$ be a polynomial of degree $2g + \epsilon$ with $2g + \epsilon$ distinct roots λ_i. Set $A := k[x, y]/(y^2 - f(x))$ and let K be the quotient field of A. Let $X := C_K$ be the non-singular projective curve with function field K. Prove that X is a hyperelliptic curve. Furthermore, the degree 2 morphism $\varphi : X \to \mathbb{P}^1$ may be chosen with the set $\{(1, \lambda_i) \ : \ 1 \leq i \leq 2g + \epsilon\}$ consisting of ramification points of φ.

   (b) Let $f(x)$ be as in part 6a, $g(x) \in k(x)$ a non-zero rational function, and set $L := k(x)[z]/(z^2 - f(x)g^2(x))$. Show that K is isomorphic to L as $k(x)$-algebras.

   (c) Let $h \in k(x)$ be a non-constant rational function. Assume that the set $B := \{Q \in \mathbb{P}^1 \ : \ \mathrm{ord}_Q(h) \text{ is odd}\}$ is non-empty. Show that the hyperelliptic curve X with function field $k(x)[y]/(y^2 - h)$ admits a morphism $\varphi : X \to \mathbb{P}^1$ of degree 2 ramified precisely over B. Hint: Reduce to the case of part 6a and show that the point at infinity $(0, 1) \in \mathbb{P}^1$ is a branch point if and only if $\epsilon = 1$.

   (d) Conclude, that the morphism φ you constructed in part 6a has precisely $2g + 2$ ramification points.

   (e) Set $Y := V\left(y^2 z^{2g+\epsilon-2} - \prod_{i=1}^{2g+\epsilon}(x - \lambda_i z)\right)$, where x, y, z are the homogeneous coordinates on $\mathbb{P}^2$. Show that there exists a birational surjective morphism $\psi : X \to Y$, where X is the curve in part 6a. Show that Y has precisely one singular point if $g > 1$, or if $g = 1$ and $\epsilon = 2$.

   (f) Show that when $g = 1$, the hyperelliptic curve in part 6a is isomorphic to a smooth plane cubic (both when $\epsilon = 1$ and when $\epsilon = 2$).

[8] **Artin's Theorem**: (see Lang's Algebra Text) Let K be a field and G a finite group of automorphisms of K of order $|G|$. Let $K^G \subset K$ be the fixed subfield. Then K is a finite Galois (i.e., normal and separable) extension of K^G, $[K : K^G] = |G|$, and $Gal(K/K^G) = G$.