Algebraic Geometry — Homework Assignment 8, — Fall 2007

Due Tuesday, November 27.

The field k below is assumed algebraically closed.

1. (Hartshorne, Problem I.5.6, *Blowing up curve singularities*). Let $Y = V(f)$ be an affine plane curve and $P = (a, b)$ a point of $\mathbb{A}^2$. Write $f = f_\mu + f_{\mu+1} + \ldots + f_d$, where f_i is a homogeneous polynomial of degree i in $(x-a)$ and $(y-b)$, and $f_\mu \neq 0$. Recall that the multiplicity of P on Y is μ. If $\mu > 0$, the tangent directions are cut out by the linear factors of f_μ. A *double point* is a point of multiplicity 2. We define a *node* (also called an *ordinary double point*) to be a double point with distinct tangent directions. Denote by $\varphi : \widetilde{Y} \to Y$ the morphism of blowing-up $P \in Y$.

   (a) Let Y be the cuspidal curve $V(y^2 - x^3)$ or the nodal curve $V(x^6 + y^6 - xy)$ from Homework 7 question 6. Show that the curve $\widetilde{Y}$, obtained by blowing up Y at the point $O := (0, 0)$, is non-singular. Note: The term *cusp* is defined in Exercise I.5.14 part d in Hartshorne. It is characterised also as a double point planar singularity $p \in Y$, such that $\varphi^{-1}(P)$ consists of a single point $\widetilde{P} \in \widetilde{Y}$ and $\widetilde{Y}$ is non-singular at $\widetilde{P}$.

   (b) Let P be a node on a plane curve Y. Show that $\varphi^{-1}(P)$ consists of two distinct non-singular points on $\widetilde{Y}$. We say that "blowing-up P resolves the singularity at P".

   (c) Let $P = (0, 0)$ be the tacnode of $Y = V(x^4 + y^4 - x^2)$ from Homework 7 question 6. Show that $\varphi^{-1}(P)$ is a node. Using 1b we see that the tacnode can be resolved by two succesive blowing-up.

   (d) Let Y be the plane curve $V(y^3 - x^5)$, which has a higher order cusp at O. Show that O is a triple point; that blowing-up O gives rise to a double point, and that one further blowing-up resolves the singularity.

2. (Hartshorne, Problem I.5.7) Let $Y \subset \mathbb{P}^2$ be a non-singular plane curve of degree > 1, defined by the equation $f(x, y, z) = 0$. Let $X \subset \mathbb{A}^3$ be the affine variety defined by f (this is the cone over Y). Let $P = (0, 0, 0)$ be the vertex of the cone and $\varphi : \widetilde{X} \to X$ the blowing-up of X at P.

   (a) Show that P is the only singular point of X.

   (b) Show that $\widetilde{X}$ is non-singular (cover it with open affine subsets).

   (c) Show that $\varphi^{-1}(P)$ is isomorphic to Y.

3. (Hartshorne, Problem I.5.8)

   (a) (Euler's Lemma) Let f be a homogeneous polynomial of degree m in the variables $x_0, \ldots, x_n$. Show that $\sum_{i=0}^{n} x_i \left(\frac{\partial f}{\partial x_i}\right) = m \cdot f$. Conclude, in particular, that if $char(k) = 0$ or does not divide m, and the partials $\frac{\partial f}{\partial x_i}$, $0 \leq i \leq n$, all vanish at a point $P \in \mathbb{P}^n$, then P belongs to $V(f)$.

(b) Let $Y \subset \mathbb{P}^n$ be a projective variety of dimension r. Let $f_1, \dots, f_t \in S = k[x_0, \dots, x_n]$ be homogeneous polynomials which generate $I(Y)$. Let $P = (a_0, \dots, a_n)$ be a point of Y. Show that P is a non-singular point of Y, if and only if the rank of the matrix $\left(\frac{\partial f_i}{\partial x_j}(a_0, \dots, a_n)\right)$ is $n - r$. Hint:

   i. Show that this rank is independent of the homogeneous coordinates chosen for P.

   ii. Pass to an open affine $U_i \subset \mathbb{P}^n$ containing P and use the affine Jacobian matrix.

   iii. Use part 3a.

4. (a) Let $f, g \in k[x_0, x_1, x_2]$ be homogeneous polynomial of positive degree. Assume that both f and g vanish at the point $P \in \mathbb{P}^2$. Set $h := fg$. Prove that $\frac{\partial h}{\partial x_i}(P) = 0$, for $0 \leq i \leq 2$.

   (b) (Hartshorne, Problem I.5.9) Let $f \in k[x_0, x_1, x_2]$ be a homogeneous polynomial, $Y := V(f) \subset \mathbb{P}^2$ the algebraic set defined by f, and suppose that for every $P \in Y$ we have $\frac{\partial f}{\partial x_i}(P) \neq 0$, for some i. Show that f is irreducible, and hence that Y is a non-singular variety). Hint: Use problem 8 in Homework 5.

   (c) (Hartshorne, Problem I.5.5) For every degree $d > 0$, and for every $p = 0$ or a prime number, give the equation of a non-singular curve of degree d in $\mathbb{P}^2$ over a field k of characteristic p.

5. (Hartshorne, Problem I.5.12 part c) Assume that $char(k) \neq 2$, and let $Q := V(f) \subset \mathbb{P}^n$, where $f(x_0, \dots, x_n) = x_0^2 + \cdots + x_r^2$, $2 \leq r \leq n$. Recall that any irreducible homogeneous polynomial of degree 2 is equivalent to such an f, after a suitable linear change of variables (Homework 3 Question 3). Show that Q is non-singular, if $r = n$, and the singular locus $Sing(Q)$ is a $\mathbb{P}^{n-r-1}$ linearly embedded in $\mathbb{P}^n$, if $r < n$.

6. (Hartshorne, Problem I.5.15 part b, modified) Let $S := k[X, Y, Z]$, and denote by $\mathcal{H}(d, 2) := \mathbb{P}S_d$ the parameter variety of all curves of degree d in $\mathbb{P}^2$, as in Homework 7 Question 5 and Homework 5 Question 5. Note that $\mathcal{H}(d, 2)$ is isomorphic to $\mathbb{P}^N$, $N = \binom{d+2}{2} - 1$. Show that the irreducible non-singular curves of degree d correspond to the points of a non-empty Zariski open subset of $\mathcal{H}(d, 2)$.
Hint: Let $F(X, Y, X, T_0, \dots, T_N)$ be the defining bi-homogeneous equation of the universal curve $\mathcal{C}$ in $\mathbb{P}^2 \times \mathcal{H}(d, 2)$, as in HW5 Q5. Consider the bi-homogeneous polynomials $\frac{\partial F}{\partial X}$, $\frac{\partial F}{\partial Y}$, $\frac{\partial F}{\partial Z}$ and use the completeness of $\mathbb{P}^2$ (Mumford, section I.9 Theorem 1), together with questions 3 and 4. Note: The results of questions 4 and 6 generalize for hypersurfaces in $\mathbb{P}^n$, using the same argument.

7. *Blowing-up points of projective varieties*: We defined in class the blowing-up $\varphi : \widetilde{Y} \to Y$ of a point P in any variety Y. Here you will show that if Y is projective then $\widetilde{Y}$ is projective.

   (a) Let $n \geq 1$, P the point $(1, 0, \dots, 0)$ in $\mathbb{P}^n$, and $\pi : \mathbb{P}^n \setminus \{P\} \to \mathbb{P}^{n-1}$ the projection, given by $\pi(a_0, \dots, a_n) = (a_1, \dots, a_n)$, as in Homework 4 question

2. Let $x_0, \dots, x_n$ be the homogeneous coordinates of $\mathbb{P}^n$ and $y_1, \dots, y_n$ those of $\mathbb{P}^{n-1}$. Prove the following statements (reduce to the affine case).

i. The closure X of the graph of π in $\mathbb{P}^n \times \mathbb{P}^{n-1}$ is equal to $V(J)$, where J is the bi-homogeneous ideal generated by $x_i y_j - x_j y_i$, $1 \leq i, j \leq n$.

ii. The restriction $\varphi : X \to \mathbb{P}^n$ of the first projection restricts to an isomorphism $X \setminus \varphi^{-1}(P) \to \mathbb{P}^n \setminus \{P\}$.

iii. The second projection $\psi : X \to \mathbb{P}^{n-1}$ restricts to an isomorphism from $\varphi^{-1}(P)$ onto $\mathbb{P}^{n-1}$.

iv. X is a closed and non-singular subvariety (irreducible) of $\mathbb{P}^n \times \mathbb{P}^{n-1}$.

Definition: Given a subvariety Y of $\mathbb{P}^n$ containing the point P, let $\widetilde{Y}$ be the closure in X of $\varphi^{-1}(Y \setminus \{P\})$. Denote by $\varphi : \widetilde{Y} \to Y$ also the restriction of the morphism φ. Then $\widetilde{Y}$ is the blowing-up of Y at P.

(b) Find bihomogeneous equations for the blow-up $\widetilde{C} \subset \mathbb{P}^2 \times \mathbb{P}^1$ of the point $(1, 0, 0)$ on $C := V(y^2 x - z^2(x + z)) \subset \mathbb{P}^2$. Show that $\widetilde{C}$ is a non-singular projective curve and the second projection $\psi : \widetilde{C} \to \mathbb{P}^1$ is an isomorphism.

8. (a) Let X be a compact Riemann surface and f a non-zero element of its function field $K(X)$. Prove that $\text{ord}_P(f) = 0$, for all but finitely many points of X. Define the *degree*[1] $\deg(f)$ of f as the sum $\sum_{\{P \in X \,:\, \text{ord}_P(f) > 0\}} \text{ord}_P(f)$ of all positive valuations of f. Show that $\deg : K(X) \setminus \{0\} \to \mathbb{Z}$ is a homomorphism from the multiplicative group of non-zero rational functions to the integers.

(b) *Automorphisms of* $\mathbb{P}^1$ (Hartshorne, section I.6 problem 6.6 page 47). Think of $\mathbb{P}^1$ as $\mathbb{A}^1 \cup \{\infty\}$. Then we define a fractional linear transformation of $\mathbb{P}^1$ by sending $x \mapsto (ax + b)/(cx + d)$, for $a, b, c, d \in k$, $ad - bc \neq 0$.

i. Show that a fractional linear transformation induces an automorphism of $\mathbb{P}^1$. We denote the group of oll these fractional linear transformations by $PGL(2)$.

ii. Let $\text{Aut}(\mathbb{P}^1)$ denote the group of all automorphisms of $\mathbb{P}^1$. Show that $\text{Aut}(\mathbb{P}^1)$ is isomorphic to $\text{Aut}(k(x))$, the group of all automorphisms of $k(x)$ as a k-algebra.

iii. Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $PGL(2) \to \text{Aut}(\mathbb{P}^1)$ is an isomorphism. See Example II.7.1.1 in Hartshorne for the generalization to the case of $\mathbb{P}^n$. Hint: Note that the homomorphism $\deg : k(x) \setminus \{0\} \to \mathbb{Z}$ is $\text{Aut}(\mathbb{P}^1)$-invariant.

9. Hartshorne, section I.6 problem 6.7 page 47. Let $P_1, \dots, P_r$, $Q_1, \dots, Q_s$ be distinct points of $\mathbb{A}^1$. Show that if $\mathbb{A}^1 \setminus \{P_1, \dots, P_r\}$ is isomorphic to $\mathbb{A}^1 \setminus \{Q_1, \dots, Q_s\}$, then $r = s$. Is the converse true?

[1] In the next homework assignment, $\deg(f)$ will be shown to be equal to the degree of the morphism $X \to \mathbb{P}^1$ induced by f.