Algebraic Geometry Homework Assignment 7, Fall 2007

Due Thursday, November 15.

The field k below is assumed algebraically closed.

1. Show that if X and Y are complete varieties, then $X \times Y$ is a complete variety.

2. Let $\phi : V_1 \to V_2$ be a morphism from a complete variety V_1 to a variety V_2, X a closed subset of V_1, $Y := \phi(X)$, and $f : X \to Y$ the restriction of ϕ. (The set-up is clumsy, since we have not defined yet morphisms from arbitrary closed subsets of varieties). Assume that i) Y is irreducible, and ii) all fibers of f are irreducible and of the same dimension d. Prove that X is irreducible.
   Note: Compare with Problem 4 in Homework 5. Hint: Let $X_1, \ldots, X_t$ be the irreducible components of X. Prove first that $f(X_i) = Y$ for some X_i. Next prove that we may choose such X_i of dimension $d + \dim(Y)$. Finally prove that if $f(X_i) = Y$ and $\dim(X_i) = d + \dim(Y)$, then $X = X_i$.

3. Construction of the Grasmannian variety $G(r, n)$: Let V be an n-dimensional vector space over k and $\overset{r}{\wedge} V$ its exterior product. Recall that $\dim\left(\overset{r}{\wedge} V\right) = \binom{n}{r}$.
   If $\{e_1, \ldots, e_n\}$ is a basis for V, then $\overset{r}{\wedge} V$ has the basis
$$\{e_{i_1} \wedge \cdots \wedge e_{i_r} \ : \ \text{where } i_1 < \cdots < i_r \text{ and } 1 \leq i_j \leq n\}. \qquad (1)$$
   Let $G(r, n)$ be the set of r dimensional subspaces of V. Consider the set theoretic map
$$[\bullet] \ : \ G(r, n) \quad \longrightarrow \quad \mathbb{P}\left(\overset{r}{\wedge} V\right) \cong \mathbb{P}^{\binom{n}{r}-1}$$
   sending an r-dimensional subspace W of V to the point $[W] \in \mathbb{P}\left(\overset{r}{\wedge} V\right)$, corresponding to the line $\overset{r}{\wedge} W$ in $\overset{r}{\wedge} V$. The basis (1) introduces homogeneous coordinates on $\mathbb{P}\left(\overset{r}{\wedge} V\right)$, called *Plücker coordinates*. The Plücker coordinates of $[W]$ can be computed in terms of a basis $\{f_1, \ldots, f_r\}$ of W as the coefficients on the right hand side of the following equation
$$f_1 \wedge \cdots \wedge f_r \quad = \sum_{1 \leq i_1 < \cdots < i_r \leq n} p[i_1, \ldots, i_r] e_{i_1} \wedge \cdots \wedge e_{i_r}.$$
   Note that $p[i_1, \ldots, i_r]$ is an $r \times r$ minor of the matrix, whose columns are the coordinate vectors of $f_1, \ldots, f_r$ in the chosen basis for V. A non-zero vector in $\overset{r}{\wedge} V$ is called *decomposeable*, if it is of the form $f_1 \wedge \cdots \wedge f_r$, for some r independent vectors in V. Denote by $D(r, n) \subset \mathbb{P}\left(\overset{r}{\wedge} V\right)$ the subset of all lines spanned by decomposable vectors. Clearly $D(r, n)$ is equal to the image of $[\bullet]$.

   (a) Let $t \in \overset{r}{\wedge} V$ be a non-zero vector and $\varphi_t : V \to \overset{r+1}{\wedge} V$ the linear homomorphism sending $x \in V$ to $t \wedge x$. Prove that t is decomposable, if and only if $\dim\ker(\varphi_t) \geq r$. Hint: If $\dim\ker(\varphi_t) \geq r$, we may choose the basis for V so that $e_i \in \ker(\varphi_t)$, for $1 \leq i \leq r$.

(b) Prove that the map $[\bullet] : G(r,n) \to D(r,n)$ is bijective. We identify the two sets from now on and denote both by $G(r,n)$.

(c) Prove that $G(r,n)$ is a Zariski closed subset of $\mathbb{P}\left(\overset{r}{\wedge} V\right)$.

(d) Let $L_0 := \text{span}\{e_1, \ldots, e_r\}$ and consider the map $q : GL(n,k) \to G(r,n)$ given by $T \mapsto T(L_0)$. Show that q is a surjective map and a morphism. Hint: Explicitly describe the Plücker coordinates of $q(T)$ in terms of the first r columns of the invertible matrix T.

(e) Prove that $G(r,n)$ is an irreducible projective variety of dimension $r(n-r)$.

(f) Let $U_{[i_1,\ldots,i_r]} \subset \mathbb{P}\left(\overset{r}{\wedge} V\right)$ be the open subset where the Plücker coordinate $p[i_1,\ldots,i_r]$ does not vanish. Prove that $G(r,n) \cap U_{[i_1,\ldots,i_r]}$ is isomorphic to $\mathbb{A}^{r(n-r)}$. Hint: Let $A \subset GL(n)$ be the subgroup consisting of matrices of the form $\begin{pmatrix} I_r & 0 \\ * & I_{n-r} \end{pmatrix}$, where I_r is the $r \times r$ identity matrix. Show that q restricts as an isomorphism from A onto $G(r,n) \cap U_{[1,\ldots,r]}$.

4. (a) Let V be a $(2k+\epsilon)$-dimensional vector space, where $\epsilon = 0$ or 1, and $t \in \overset{2}{\wedge} V$. A standard fact from linear algebra states that there exists a basis $\{e_1, \ldots, e_{2k+\epsilon}\}$ of V, with respect to which $t = \sum_{i=1}^{k} c_i e_{2i-1} \wedge e_{2i}$. I) Prove that anti-symmetric bilinear forms have *even* rank. II) Given a $2k$-dimensional vector space V and an element $t \in \overset{2}{\wedge} V$, denote by $T : V^* \to V$ the anti-self-dual linear transformation induced by t. The polynomial map $P : \overset{2}{\wedge} V \to \overset{2k}{\wedge} V$, given by $t \mapsto t^k$, is an element of $Sym^k(\overset{2}{\wedge} V)^* \otimes \overset{2k}{\wedge} V$. More explicitly, if we choose coordinates on V, then P is a polynomial of degree k in the coordinates of $\overset{2}{\wedge} V$, called the Pffafian. On the other hand, $\det(T) := \overset{2k}{\wedge} T$ belongs to $Sym^{2k}(\overset{2}{\wedge} V)^* \otimes (\overset{2k}{\wedge} V)^{\otimes 2}$, i.e., $\det : \overset{2}{\wedge} V \to (\overset{2k}{\wedge} V)^{\otimes 2}$ is a polynomial of degree $2k$ in the coordinates of $\overset{2}{\wedge} V$. Prove that the determinant is equal to a universal non-zero constant times the square of the Pffafian.

(b) Show that a vector $t \in \overset{2}{\wedge} V$ is decomposable, if and only if $t \wedge t = 0 \in \overset{4}{\wedge} V$.

(c) Prove that $G(2,4)$ is a quadric hypersurface in $\mathbb{P}^5$ and find its homogeneous quadratic equation in the Plücker coordinates.

(d) Let $Q(x_0, \ldots, x_5)$ be a quadratic polynomial with a non-degenerate symmetric bilinear form. Prove that the quardic hypersurface $V(Q)$ in $\mathbb{P}^5$ is isomorphic to $G(2,4)$. Hint: See problem 3 in Homework 3.

5. Assume now that V is $n+1$ dimensional so that $\mathbb{P}V$ is isomorphic to $\mathbb{P}^n$. Choose homogeneous coordinates on $\mathbb{P}V$, let $S = k[x_0, \ldots, x_n]$ be the homogeneous coordinate ring of $\mathbb{P}V$, and let S_d be its graded summand of degree d. Set $\mathcal{H}(d,n) := \mathbb{P}S_d$. A point in $\mathcal{H}(d,n)$ parametrizes a hypersurface of degree d in $\mathbb{P}^n$. Let

$$I(r,n,d) \quad \subset \quad \mathcal{H}(d,n) \times G(r+1, n+1)$$

be the incidence subset, consisting of pairs (X,W), such that the r-dimensional linear subspace $\mathbb{P}W$ of $\mathbb{P}^n$ is contained in the hypersurface X. One easily checks that $I(r,n,d)$ is a Zariski closed subset of $\mathcal{H}(d,n) \times G(r+1, n+1)$.

(a) Show that the projection $p_2 : I(r,n,d) \to G(r+1,n+1)$ is surjective and its fiber over $W \in G(r+1,n+1)$ is a linear subspace of $\mathcal{H}(d,n)$ of dimension $\binom{n+d}{d} - \binom{r+d}{d} - 1$. Hint: Identify S_d with $\mathrm{Sym}^d V^*$ and consider the natural restriction homomorphism $\mathrm{Sym}^d V^* \to \mathrm{Sym}^d W^*$.

(b) Prove that $I(r,n,d)$ is an irreducible variety of dimension $(r+1)(n-r) + \binom{n+d}{d} - \binom{r+d}{d} - 1$. Hint: Consider Problem 2

(c) Prove that the image of the first projection $p_1 : I(r,n,d) \to \mathcal{H}(d,n)$ is a closed subvariety of $\mathcal{H}(d,n)$. Hint: A one line argument!

(d) Assume that $(n-r)(r+1) < \binom{r+d}{d}$. Prove that $p_1(I(r,n,d))$ is a proper subset of $\mathcal{H}(d,n)$. Conclude that for $d \geq 4$, there is a dense open subset $\mathcal{H}'(d,3)$ in $\mathcal{H}(d,3)$, such that for $X \in \mathcal{H}'(d,3)$, the corresponding surface X of degree d in $\mathbb{P}^3$ does not contain any line.

(e) Show that every cubic surface in $\mathbb{P}^3$ contains a line. Hint: Set $n=3$, $r=1$, and $d=3$ and note that $\dim I(1,3,3) = \dim \mathcal{H}(3,3)$. Show first that the (singular) cubic $x_0x_1x_2 - x_3^3$ contains only 3 lines.

(f) Find 27 lines on the Fermat cubic surface $V(x_0^3 + x_1^3 + x_2^3 + x_3^3) \subset \mathbb{P}^3$.

Note: It can be proven that over the open subset of $\mathcal{H}(3,3)$, where X is smooth, the fiber $p_1^{-1}(X)$ consists of 27 points; representing 27 lines on X.

6. (Hartshorne Exercise I.5.1) Locate the singular points of the following curves in $\mathbb{A}^2$ (assume that the characteristic of k is not equal to 2). a) $x^2 = x^4 + y^4$, b) $xy = x^6 + y^6$, c) $x^3 = y^2 + x^4 + y^4$, d) $x^2y + xy^2 = x^4 + y^4$. Sketch these curves when $k = \mathbb{R}$. A scketch is provided in Hartshorne.

7. (Hartshorne Exercise I.5.2) Locate the singular points and describe the singularities of the following surfaces in $\mathbb{A}^2$. a) $xy^2 = z^2$, b) $x^2 + y^2 = z^2$, c) $xy + x^3 + y^3 = 0$. A scketch is provided in Hartshorne.