Algebraic Geometry Homework Assignment 6, Fall 2007
Due Tuesday, November 6.

The field k below is assumed algebraically closed.

1. Mumford Proposition 5 page 43: Let X and Y be varieties. Prove the equality $\dim(X \times Y) = \dim(X) + \dim(Y)$.

2. (a) Let Y be a closed subvariety of an affine variety X. Let R and S be the coordinate rings of X and Y. Set $N^* := I(Y)/I(Y)^2$. Show that N^* is an S-module. Given a point $Q \in Y$ with maximal ideals $m_Q \subset R$ and $\bar{m}_Q \subset S$, check that we have an isomorphism $N^*/\bar{m}_Q N^* \cong I(Y)/m_Q I(Y)$ of k-vector spaces. $N^*/\bar{m}_Q N^*$ is called the *conormal space* at Q to Y in X.

   (b) Show that if $I(Y)$ is generated by r elements $\{f_1, \ldots, f_r\} \subset R$, then for every point $Q \in Y$, we have the inequality $\dim_k[I(Y)/m_Q I(Y)] \leq r$.
   Note: If $X = \mathbb{A}^n$, or more generally if X is smooth, and the number of generators r of $I(Y)$ is equal to codim(Y in X), then Y is said to be a *complete intersection.* In that case, $I(Y)/I(Y)^2$ is a free S module of rank r (to be proven later in the course).

   (c) Hartshorne Excercise I.1.11 (modified): Let $Y \subset \mathbb{A}^3$ be the image of the morphism $\varphi : \mathbb{A}^1 \to \mathbb{A}^3$ given by $\varphi(t) = (t^3, t^4, t^5)$.

      i. Show that Y is a closed and irreducible subvariety of $\mathbb{A}^3$ and $\dim(Y) = 1$.

      ii. Show that the ideal $I(Y)$ can not be generated by two elements. Hint: Consider the coordinate algebra $A := k[x, y, z]$ of $\mathbb{A}^3$ as a graded algebra with $\deg(x) = 3$, $\deg(y) = 4$, and $\deg(z) = 5$. Then $\varphi^* : A \to k[t]$ is a graded homomorphism of graded rings. Conclude that $I(Y)$ is a graded ideal. Let $A_{\leq d}$ be the subspace of polynomials of weighted degree $\leq d$. Calculate $\dim(A_{\leq 10} \cap I(Y))$ and $\dim(A_{\leq 10} \cap m_0 I(Y))$.

3. (Mumford, the last problem in section I.6) Let $Y \subset \mathbb{P}^n$ be defined by a homogeneous prime ideal $P \subset k[X_0, \ldots, X_n]$. Let $Y^* \subset \mathbb{A}^{n+1}$ be the affine cone over Y (the affine variety defined by P). Denote by Y_{X_i} the subset of Y, where $X_i \neq 0$, and let $Y^*_{X_i}$ be the subset of Y^*, where $X_i \neq 0$. Show that $Y^*_{X_i}$ is isomorphic to $(\mathbb{A}^1 \setminus \{0\}) \times Y_{X_i}$. Note: We used this fact in the proof of Theorem 2^* in section I.7.

4. Let $gl(n, k)$ be the variety of $n \times n$ matrices with entries in the field k, and let $char : gl(n, k) \to \mathbb{A}^n$ be the morphism, which takes a matrix A to the coefficients $(a_1, \ldots, a_n)$ of its characteristic polynomial $x^n + a_1 x^{n-1} + \cdots + a_n$.

   (a) Prove that *char* is a surjective morphism, and that all its fibers are of pure dimension $n^2 - n$. Hint:[1] Use the k^*-action on $gl(n, k)$ and the Upper-Semi-Continuity Theorem for fiber dimension, Mumford section I.8 Corollary 3, to reduce the question to the nilpotent case.

[1]The intention is for you to prove that the fibers are of pure-dimension $n^2 - n$ in two ways; the one hinted here, as well as again in part 4b in the course of proving irreducibility.

(b) Prove that all fibers of *char* are irreducible. Hint: Show first that each fiber is a union of finitely many $GL(n,k)$ orbits, each of which is irreducible, and precisely one of them is (n^2-n)-dimensional. You are likely to find the following standard linear algebra result helpful.

Theorem: (Jordan Decomposition Theorem) Every element $A \in gl(n,k)$ admits a unique decomposition $A = D + N$ satisfying

i. D is a diagonalizable matrix and N is a nilpotent matrix.

ii. There exists polynomials $d(x), n(x) \in k[x]$, such that $D = d(A)$ and $N = n(A)$. In particular, $DN = ND$.

5. (Springer fibers of type A_2) Let $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ be $V(\sum_{i=0}^{2} X_i Y_i)$, where X_0, X_1, X_2; Y_0, Y_1, Y_2 are the bi-homogeneous coordinates on $\mathbb{P}^2 \times \mathbb{P}^2$ (see problem 2 in Homework 5). X is the full flag variety $Flag(1,2,3)$, if we regard a point $a := (a_0, a_1, a_2)$ in the first factor as a vector, and a point $b := (b_0, b_1, b_2)$ in the second factor as a linear functional, both up to a scalar factor, so that a point $(a,b) \in X$ corresponds to the flag $\mathrm{span}\{a\} \subset \ker(b) \subset k^3$. Equivalently, X is the *incidence variety* of pairs of a point and a line containing it. Set $\tilde{a} := \mathrm{span}\{a\}$ and $\tilde{b} := \ker(b)$. Denote by $sl(3,k)$ the affine space of 3×3 traceless matrices with entries in k. Let $Y \subset sl(3,k) \times X$ be the subset

$$Y \;:=\; \left\{(M,a,b) \;:\; M(\tilde{a}) = (0),\; M(\tilde{b}) \subset \tilde{a},\; \text{and } \mathrm{Im}(M) \subset \tilde{b}\right\}$$

and $\pi_1 : Y \to sl(3,k)$ the first projection.

(a) Prove that Y is an irreducible algebraic subset of dimension 6. Hint: See Problems 4 and 5 in Homework 5. Note: Y is the *cotangent bundle* of X.

(b) Show that the image of π_1 is the fiber $char^{-1}(0)$.

(c) Show that the isomorphism class of the fiber $\pi_1^{-1}(A)$ depends only on the similarity class of A. Conclude that there are three types of fibers. Hint: Show that $GL(3,k)$ acts on $sl(3,k) \times \mathbb{P}^2 \times \mathbb{P}^2$ by $T(M,a,b) = (TMT^{-1}, T(a), bT^{-1})$, and Y is $GL(3,k)$-invariant.

(d) Describe the fibers $\pi_1^{-1}(0)$ and $\pi_1^{-1}(B)$, where $B := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Conclude, that π_1 is a birational morphism onto $char^{-1}(0)$.

(e) Set $C := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Show that the fiber $\pi_1^{-1}(C)$ is reducible, with two irreducible components, each isomorphic to $\mathbb{P}^1$. Now interpret your answer conceptually. Show more generally, that if M is a nilpotent matrix of rank 1, then the fiber $\pi_1^{-1}(M)$ is naturally isomorphic to the union of the two copies $\mathbb{P}[k^3/Im(M)]$ and $\mathbb{P}[\ker(M)]$ of $\mathbb{P}^1$ meeting at one point corresponding to the flag $Im(M) \subset \ker(M) \subset k^3$.