Algebraic Geometry Homework Assignment 5, Fall 2007

Due Thursday, October 27.

The field k below is assumed algebraically closed.

1. Let $f : \mathbb{A}^2 \to \mathbb{A}^2$ be the morphism given by $f(x, y) = (xy, y)$. Describe the image of f. Show that f fails to satisfy each of properties 1 and 2 in Proposition 2 of section I.7 in Mumford (it is not closed, and it has an infinite fiber).

2. (Fulton's *Algebraic curves* problem 4.28 modified) Set $A := k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$. A polynomial $F \in A$ is called *bi-homogeneous* of bi-degree (p, q), if F is homogeneous of degree p (resp. q) when considered as a polynomial in $X_0, \ldots, X_n$ (resp. in the Y_i's). Given a set S of bi-homogeneous polynomials, set

$$V(S) \quad := \quad \{(x, y) \in \mathbb{P}^n \times \mathbb{P}^m \ : \ F(x, y) = 0, \text{ for all } F \in S\}.$$

   (a) Show that a subset Z of $\mathbb{P}^n \times \mathbb{P}^m$ is closed, in the Zariski topology of the product variety defined in Mumford section 6, if and only if $Z = V(S)$, for some set S of bi-homogeneous polynomials.

   (b) Let $A_{++} \subset A$ be the ideal generated by all the products X_iY_j, $0 \leq i \leq n$, $0 \leq j \leq m$. Prove the bi-homogeneous Nullstellensatz: There is a one-to-one order reversing correspondence between radical bi-homogeneous ideals not containing A_{++}, and non-empty closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$. Hint: Immitate the proof that affine Nallstellensatz implies projective Nallstellensatz (Theorem 3 in section I.2 of Mumford's text). Note that the subset $V_{\text{affine}}(A_{++})$ of $\mathbb{A}^{n+m+2}$ is the union $\mathbb{A}^{n+1} \times \{0\} \cup \{0\} \times \mathbb{A}^{m+1}$. Use the linear algebra fact, that an ideal of A is bi-homogeneous, if and only if it is $(k^* \times k^*)$-invariant.

   (c) Assume that $V(S) \neq \emptyset$. Show that $V(S)$ is irreducible, if and only if the bi-homogeneous ideal, generated by S, is a prime ideal in A.

   (d) Let $\varphi : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ be the Segre embedding,

$$\varphi[(x_0, x_1), (y_0, y_1)] = (x_0y_0, x_0y_1, x_1y_0, x_1y_1).$$

   Let the homogeneous coordinates on $\mathbb{P}^3$ be X, Y, Z, W, so that the image of φ is $V(XW - YZ)$.

      i. Find a bi-homogeneous polynomial $F(X_0, X_1, Y_0, Y_1)$, such that $\varphi(V(F)) = V(XW - YZ,\ X^2 + Y^2 + Z^2 + W^2)$. Show that $V(F)$ is the union of four irreducible components, each isomorphic to $\mathbb{P}^1$.

      ii. Let $\rho : \mathbb{P}^1 \to \mathbb{P}^3$ be the twisted cubic, $\rho(s, t) = (s^3, s^2t, st^2, t^3)$. Find a bi-homogeneous polynomial $G(X_0, X_1, Y_0, Y_1)$, such that $\varphi^{-1}(\rho(\mathbb{P}^1)) = V(G)$. Compare with problem 5 of Homework 2.

3. Let $Z \subset \mathbb{P}^n \times \mathbb{P}^m$ be a closed subvariety, $I(Z)$ its bi-homogeneous prime ideal, $\Gamma_b(Z) := A/I(Z)$, and $K_b(Z)$ its quotient field. Set $K'(Z)$ to be $\{0\}$ union

$$\left\{ h \in K_b(Z) \ ; \ h = \frac{F}{G},\ F,\ G \text{ are bi-homogeneous of the same bi-degree in } \Gamma_b(Z) \right\}.$$

   Show that $K'(Z)$ is isomorphic to the function field $K(Z)$ of Z. Describe the local ring $\mathcal{O}_z$, at a point $z \in Z$, in terms of $\Gamma_b(Z)$.

4. Let X and Y be varieties, C a closed subset of X, and $f : C \to Y$ a continuous surjective map (this is the case, for example, if f is the restriction of a morphism from X to Y). Assume that i) every fiber of f is an irreducible subset of C, and ii) For every $y \in Y$ there exists a Zariski open subset U of Y, containing y, such that $f^{-1}(U)$ is homeomorphic to $U \times f^{-1}(y)$. Prove that C is irreducible.
   Note: The latter condition states that $f : C \to Y$ is (topologically) a locally trivial fibration.

5. (Fulton's *Algebraic curves* problem 6.28) Let $d \geq 1$, $N = \frac{(d+1)(d+2)}{2} - 1$, and let $M_0, \ldots, M_N$ be the monomials of degree d in X, Y, Z (in some order). Let $T_0, \ldots, T_N$ be homogeneous coordinates for $\mathbb{P}^N$. Set

$$\mathcal{C} \;:=\; V\left(\sum_{i=0}^{N} M_i(X,Y,Z)T_i\right) \quad \subset \mathbb{P}^2 \times \mathbb{P}^N,$$

   and let $\pi : \mathcal{C} \to \mathbb{P}^N$ be the restriction of the projection map.

   (a) Show that $\mathcal{C}$ is an irreducible closed subvariety of $\mathbb{P}^2 \times \mathbb{P}^N$, and π is a morphism. Hint: Prove the irreducibility in two ways i) Using Problem 2c. ii) Consider the fibers of the other projection $p : \mathcal{C} \to \mathbb{P}^2$ and use Problem 4.

   (b) For each $t = (t_0, \ldots, t_N) \in \mathbb{P}^N$, set $F_t := \sum_{i=0}^N t_i M_i(X,Y,Z) \in k[X,Y,Z]$, and $C_t := V(F_t) \subset \mathbb{P}^2$. Show that $\pi^{-1}(t) = C_t \times \{t\}$.

   Note: The data of the projective variety $\mathcal{C}$, together with the morphism $\pi : \mathcal{C} \to \mathbb{P}^N$, is called the *universal family* of curves of degree d. If the polynomial F_t is square free, so that (F_t) is a radical ideal, then $\deg(C_t)$ is defined to be $\deg(F_t)$, which is d. If one of the irreducible factors of F_t appears with multiplicity $\mu > 1$, then the algebraic set C_t will have degree $< d$, but we will define later a natural *scheme structure* on the fiber $\pi^{-1}(t)$, which encodes these multiplicities.

6. (a) Let $f : X \to Y$ be a morphism of varieties, V an open subset of Y and U an open subset of X, which is mapped into V. Prove that U and V are varieties, and f resticts to a morphism from U to V. Hint: Use Proposition 6 section I.5 in Mumford's text. Note: Consider the special case, where Y is affine and $X = U$. The coordinate ring $\Gamma(Y)$ is a subring of $\Gamma(V)$. Part 6a says that it suffices to chech that $f^*\Gamma(Y)$ is contained in $\Gamma(X)$, in order to conclude that $f^*\Gamma(V)$ is contained in $\Gamma(X)$.

   (b) (Fulton's *Algebraic curves* problem 6.29) Let G be a variety, and suppose G is also a group, i.e., there are functions $\varphi : G \times G \to G$ (multiplication) and $\psi : G \to G$ (inverse) satisfying the group axioms. If φ and ψ are morphisms, G is said to be an *algebraic group*. Show that each of the following is an algebraic group.

      i. $\mathbb{A}^1 = k$, with the usual addition on k; this group is often denoted by G_a.
      ii. $\mathbb{A}^1 \setminus \{0\}$ with the usual multiplication on k; this is denoted by G_m.
      iii. $GL_n(k)$, the group of invertable $n \times n$ matrices, which is an affine open subset of $\mathbb{A}^{n^2}(k)$.

Remark: Example F in section I.3 of Mumford's text, which is revisited at the end of section I.5, describe an automorphism of order 2 of a projective cubic plane curve. This is the inversion for an algebraic group structure.

7. (a) Let X and Y be affine varieties over an algebraically closed field k with coordinate rings R and S, and $\varphi : X \to Y$ a morphism. Let $Z := \overline{\varphi(X)}$ be the closure of the image of φ in Y. Show that φ factors through an isomorphism of X onto Z, if and only if the k-algebra homomorphism $\varphi^* : S \to R$ is surjective.

   (b) Formulate and prove a necessary and sufficient criterion for a morphism $\varphi : X \to Y$ of prevarieties to factor through an isomorphism onto $Z := \overline{\varphi(X)}$. In this case, we say that φ is a *closed immersion*. See Proposition 5 section I.5 in Mumford's text for the prevariety structure of Z.

   (c) Mumford, Problem in section 6 (a converse to Proposition 6 in section I.6): Let X be a prevariety, $\{U_i\}$ an affine open covering of X. Let R_i be the coordinate ring of U_i. Assume that $U_i \cap U_j$ is an affine subset of X with coordinate ring $R_i \cdot R_j$ (the minimal k-subalgebra of the function field $K(X)$ containing R_i and R_j). Prove that X is a variety.

8. (Hartshorne, Exercise I.3.7) Let $X \subset \mathbb{P}^n$ be a projective variety. We will prove that $\mathcal{O}_X(X) = k$ (all global regular functions on X are constant, an immediate corollary of the *completeness* of X, defined and proven in section I.9 in Mumford). Use this fact, together with Problem 6 of Homework 4, to solve the following:

   (a) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

   (b) Show that any two curves in $\mathbb{P}^2$ have a non-empty intersection.

   (c) More generally, show that if $X \subset \mathbb{P}^n$ is a projective variety of dimension ≥ 1, and if $Y = V(F)$ is a hypersurface (where F is a homogeneous polynomial of positive degree), then $X \cap Y \neq \emptyset$.