Algebraic Geometry — Homework Assignment 4, — Fall 2007

Due Thursday, October 11.

The field k below is assumed algebraically closed.

1. Two examples of presheaves, which are not sheaves. Let X be the complex plane $\mathbb{C}$, with its classical topology, and $\mathcal{O}_X$ the sheaf of holomorphic functions.

   (a) Let z be the coordinate on $\mathbb{C}$. Consider $\mathcal{O}_X$ as a sheaf of vector spaces and let $\frac{\partial}{\partial z}$ be the sheaf endomorphism corresponding to differentiation
   $$\begin{array}{cccc} \frac{\partial}{\partial z}: & \mathcal{O}_X(U) & \to & \mathcal{O}_X(U) \\ & f & \mapsto & \frac{\partial f}{\partial z}. \end{array}$$
   Let $\mathcal{D}$ be the image presheaf $\mathcal{D}(U) := \frac{\partial}{\partial z}(\mathcal{O}_X(U))$. Show that $\mathcal{D}$ is a presheaf and that it satisfies the first sheaf axiom (on page 18 in Mumford's text), but fails to satisfy the second.

   (b) Let $\mathcal{Q}$ be the co-kernel presheaf $\mathcal{Q}(U) := \mathcal{O}_X(U)/\mathcal{D}(U)$, with the restriction homomorphisms induced by those of $\mathcal{O}_X$. Show that $\mathcal{Q}$ is a presheaf, but that it does not satisfy the first sheaf axiom.

   (c) Prove that $\frac{\partial}{\partial z}$ induces a surjective homomorphism on the stalks of $\mathcal{O}_X$.

   (d) Prove that the sheafification of $\mathcal{D}$ is $\mathcal{O}_X$ and the sheafification of $\mathcal{Q}$ is the zero sheaf.

2. *Projection from a point.* Let $\pi : \mathbb{P}^n \setminus \{(1, 0, \ldots, 0)\} \to \mathbb{P}^{n-1}$ be the map given by $(a_0, \ldots, a_n) \mapsto (a_1, \ldots, a_n)$. Prove that π is a morphism of prevarieties (see Section 5 Proposition 6 in Mumford).

3. Show that the global sections of $\mathcal{O}_{\mathbb{P}^n}$ are constant, $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$.

4. (a) Let $\varphi : X \to Y$ be a continuous map of topological spaces and $\mathcal{F}$ a sheaf of k-algebras on X. To each open set U on Y, define $\mathcal{G}(U) := \mathcal{F}(\varphi^{-1}(U))$. Show that $\mathcal{G}$ is a sheaf k-algebras on Y. The sheaf $\mathcal{G}$ is denoted by $\varphi_*\mathcal{F}$, and is called the *push-forward* (or *direct image*) of $\mathcal{F}$.

   (b) Let X and Y be prevarieties and $\varphi : X \to Y$ a morphism. Show that $\varphi_*\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras, i.e., that there is a homomorphism of sheaves of k-algebras $h : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$.

   (c) Let X and Y be both the affine line $\mathbb{A}^1$, and $\varphi : X \to Y$ the morphism given by $\varphi(a) = a^n$. Show that $\varphi_*\mathcal{O}_X$ is isomorphic to the direct sum $\mathcal{O}_Y \oplus \cdots \oplus \mathcal{O}_Y$ of n copies of $\mathcal{O}_Y$.

   (d) Let X and Y be both $\mathbb{P}^1$ and $\varphi : X \to Y$ the morphism given by $\varphi(s, t) = (s^2, t^2)$. Show that the stalk $(\varphi_*\mathcal{O}_X)_y$, at each point y in Y, is a free $\mathcal{O}_{Y,y}$-module of rank 2, but that $\varphi_*\mathcal{O}_X$ is *not* isomorphic to $\mathcal{O}_Y \oplus \mathcal{O}_Y$. Hint: For the latter statement, consider the global sections of both sheaves.

5. (Hartshorne, problem I.3.4) Recall the d-Uple embedding $\varphi : \mathbb{P}^n \to \mathbb{P}^N$, where $N = \binom{n+d}{d} - 1$, defined in Problem 7 of Homework 2. Show that φ is an isomorphism onto its image.

6. (Mumford, Problem in section I.5 page 32) Let $F \in k[x_0, \ldots, x_n]$ be a homogeneous polynomial of positive degree. Prove that $\mathbb{P}^n_F := \{x \in \mathbb{P}^n \ : \ F(x) \neq 0\}$ is an affine variety. Hint: Consider the d-Uple embedding with $d = \deg(F)$.