Algebraic Geometry Homework Assignment 3, Fall 2007

Due Thursday, October 4.

The field k below is assumed algebraically closed.

(1) (Hartshorne, Exercise I.3.2, two examples, that a morphism, whose underlying map on the topological spaces is a homeomorphism, need not be an isomorphism).
   (a) (Solved in Mumford's Example O page 22) Let $\varphi : \mathbb{A}^1 \to \mathbb{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a morphism and a homeomorphism (bijective) from $\mathbb{A}^1$ onto $V(y^2 - x^3)$, but that φ is not an isomorphism.
   (b) (Solved in Mumford's Example N page 22) Let the characteristic of k be a prime $p > 0$, and define a map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ by $t \mapsto t^p$. Show that the morphism φ is a homeomorphism, but not an isomorphism. This is called the *Frobenius morphism*.

(2) (Hartshorne, Exercise I.3.6, an example of a quasi-affine variety, which is not affine). Show that $X := \mathbb{A}^2 \setminus \{(0,0)\}$ is not affine. Hint: Show that $\Gamma(X) \cong k[x, y]$ and use Proposition 1 in section 3 page 14 in Mumford's text.

(3) The following problem was touched upon in class, in connection to Example D of section 3 in Mumford's text. Assume that the characteristic $char(k)$ is different from 2. Let $f \in k[x_0, \dots, x_n]$ be a homogeneous polynomial of degree 2. By the theory of symmetric bilinear forms, there is a linear change of variables, which brings f to the form $x_0^2 + \cdots + x_k^2$, for some $0 \leq k \leq n$ (see Hoffman and Kunze, *Linear Algebra*, for example).
   (a) Show that f is irreducible, if and only if $k \geq 2$.
   (b) Show that after a linear change of coordinates, every plane conic (i.e., $V(f) \subset \mathbb{P}^2$, where f is irreducible, of degree 2, and $n = 2$) can be realized as the image $V(xz - y^2)$ of the 2-uple embedding $\phi : \mathbb{P}^1 \to \mathbb{P}^2$, given by $(s, t) \mapsto (s^2, st, t^2)$ (see Homework 2 Problem 7).
   (c) Construct an embedding $e : PGL(2) \to PGL(3)$, obtaining an action of $PGL(2)$ on $\mathbb{P}^2$, with respect to which the map ϕ is $PGL(2)$-equivariant, i.e., such that $\phi(g(s, t)) = e(g)\phi(s, t)$, for all $(s, t) \in \mathbb{P}^1$.
   (d) Let $C := V(f) \subset \mathbb{P}^2$ be an irreducible conic and $P = (a_0, a_1, a_2)$ a point in C. Let f_x be the partial $\frac{\partial f}{\partial x}$. Show that the line
$$f_x(P)x + f_y(P)y + f_z(P)z = 0$$
   intersects C at the point P and at no other point, and that any other line in $\mathbb{P}^2$ through P intersects C at precisely one additional point. Hint: $PGL(2)$ acts (triply) transitively on $\mathbb{P}^1$, so the statement reduces to the case $f(x, y, z) = xz - y^2$ and $P = (1, 0, 0)$.

(4) Let R be a commutative ring with 1 and S a multiplicatively closed subset. Here are two important properties of the ring of fractions $S^{-1}R$. Either work them out yourself, or look-up the proof in the literature (see for example Atiyah-MacDonald, Proposition 3.11).
   (a) Show that every ideal in $S^{-1}R$ is generated by the image of some ideal in R, via the natural homomorphism $R \to S^{-1}R$.
   (b) Show that the prime ideals of $S^{-1}R$ are in one-to-one correspondence with prime ideals of R which do not meet S. Hint: You may use the following special case of the exactness propery of the operation S^{-1}. If I is an ideal in R and $\bar{S}$ is the image of S in R/I, then $S^{-1}R/S^{-1}I \cong \bar{S}^{-1}(R/I)$, where $S^{-1}I$ is the ideal generated by the image of I.

(5) (Hartshorne, Exercise I.3.11 modified) Let X be an affine variety, $P \in X$ a point, and $m_P \subset \Gamma(X)$ its maximal ideal. Show that there is a one-to-one correspondence between the prime ideals of $\Gamma(X)_{m_p}$ and closed subvarieties of X containing P. Conclude, in particular, that $\Gamma(X)_{m_p}$ has a unique maximal ideal.

(6) Let R be a commutative ring with 1.

   (a) (Atiyah-MacDonald, Section 3 Exercise 2) Let S and T two multiplicatively closed subsets of R, and let U be the image of T in $S^{-1}R$. Show that the rings $(ST)^{-1}R$ and $U^{-1}(S^{-1}R)$ are isomorphic. Hint: This is just an elaborate use of the universal property of the rings of fractions.

   (b) Let $p \subset R$ be a prime ideal, $f \in R \setminus p$, and $\tilde{p}$ the prime ideal of R_f generated by the image of p (see Problem 4b). Prove that the rings of fractions R_p and $(R_f)_{\tilde{p}}$ are naturally isomorphic.

(7) Let R be a commutative ring with 1, $f \in R \setminus \{0\}$, $S := \{f^n \ : \ n \geq 0\}$, and $R_f := S^{-1}R$.

   (a) Set $A := R[y]/(yf - 1)$, where y is an indeterminate, and let $\phi : R \to A$ be the natural homomorphism. Prove that $\phi(r) = 0$, if and only if $rf^n = 0$, for some $n \geq 0$.

   (b) Let $h : R_f \longrightarrow A$ be the natural homomorphism, which is determined by the universal property of R_f and sends r/f^n to $\phi(r)y^n$. Prove that h is an isomorphism.

(8) Let $X \subset \mathbb{A}^n$ be an affine variety, $I(X) \subset k[x_1, \ldots, x_n]$ its ideal, and $\Gamma(X)$ its coordinate ring. In Parts 8c, 8d, and 8e below you will be filling in details left out in the proof of Proposition 4 in section 4 page 24 in Mumford. Use problems 6b and 7b, where R is not assumed to be an integral domain. This way your proof will easily adapt to a proof of a more general result, for affine schemes, which are the object of study later in the course (see Proposition 3 in section II.1 in Mumford's text).

   (a) Show that the open sets $X_f := X \setminus V(f)$, $f \in \Gamma(X)$, form a basis for the Zariski topology of X. They are called the *basic open subsets of* X.

   (b) Prove that two basic open subsets X_g and X_f satisfy $X_g \subset X_f$, if and only if $g \in \sqrt{(f)}$.

   (c) Let $f \in \Gamma(X)$ be a non-zero element, choose $F \in k[x_1, \ldots, x_n]$, such that $f = F + I(X)$, let $J \subset k[x_1, \ldots, x_n, y]$ be the ideal generated by $I(X)$ and $yF - 1$, and set $X_F := V(J)$. Prove that the affine algebraic set X_F is irreducible, and that $\Gamma(X_F)$ is isomorphic to the localization $\Gamma(X)_f$ of $\Gamma(X)$ with respect to the multiplicatively closed subset $\{f^n \ : \ n \geq 1\}$.

   (d) Let $\pi : X_F \longrightarrow X$ be the projection on the first n coordinates. Prove that π is a morphism and that its image $\pi(X_F)$ is the basic open subset X_f. Show that the map $\pi : X_F \to X_f$ is a homeomorphism.

   (e) Prove that π is an isomorphism. Hint: Use Problem 6.

(9) Let R be a commutative ring with 1 and $M \subset R$ a maximal ideal. Show that the following are equivalent:

   (a) M is the unique maximal ideal of R.

   (b) Every element of $R \setminus M$ is invertible in R.

   A ring R with the above properties is called a *local ring*. Let $I \subset k[x, y]$ be a proper ideal. Assume that there exist positive integers n and m, such that both x^n and y^m belong to I. Set $R := k[x, y]/I$, $M := (x, y)/I$, $S := R \setminus M$, and $R_M := S^{-1}R$. Show that the natural homomorphism $R \to R_M$ is an isomorphism.