Algebraic Geometry Homework Assignment 10, Fall 2007

1. (Hartshorne, Exercise II.2.2) Let $(X, \mathcal{O}_X)$ be a prescheme (in Mumford's terminology) and let $U \subset X$ be any open subset. Show that $(U, [\mathcal{O}_X]_{|U})$ is a prescheme. We call this the *induced prescheme structure* on the open set U, and we refer to $(U, [\mathcal{O}_X]_{|U})$ as an *open subscheme* of X. Compare with the analogous statement for varieties; given in Proposition 4 in section I.5 of Mumford's text.

2. (Hartshorne Excercise II.2.19, modified) Let A be a commutative ring with 1. Show that the following are equivalent:

   (a) Spec(A) is disconnected.

   (b) There exist non-zero elements $e_1, e_2 \in A$, such that $e_1e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$. These elements are called *orthogonal idempotents*.

   (c) A is isomorphic to the product $A_1 \times A_2$ of two non-zero rings.

   The above question combines with Question 3 below to yield a generalization of Homework 9 Question 1 (A here is A/I in HW9 Q1). The implication 2c$\Rightarrow$ 2a was proven in class (see also Example A in section II.1 in Mumford). Set $X := \mathrm{Spec}(A_1 \times A_2)$. We furthermore showed that the ring of fractions $(A_1 \times A_2)_{(1,0)}$, associated to $(1, 0) \in A_1 \times A_2$, is isomorphic to A_1, the distinguished open set $X_{(1,0)}$ is isomorphic to Spec(A_1), and X is the disjoint union $X_{(1,0)} \cup X_{(0,1)}$. Furthermore, given any open subset $U \subset X$, we have $\mathcal{O}_X(U) \cong \left[\mathcal{O}_{X_{(1,0)}}(U \cap X_{(1,0)}) \oplus \mathcal{O}_{X_{(0,1)}}(U \cap X_{(0,1)})\right]$. Complete the proof of the above exercise.
   Hint for the implication 2a$\Rightarrow$ 2c: Set $X := \mathrm{Spec}(A)$ and suppose that X is the disjoint union of two closed subsets X_1 and X_2. Choose radical ideals I_j, such that $X_j = V(I_j)$. Prove that the natural homomorphism $h : A \to (A/I_1) \times (A/I_2)$ is surjective and that its kernel is the nilradical of A. Conclude that each X_i is a distinguished open subset $X_i = X_{f_i}$. The initial pair $\{f_1, f_2\}$ need not consist of idempotents. Then use the sheaf axioms to prove that the natural homomorphism $A \cong \mathcal{O}_X(X) \to \mathcal{O}_X(X_{f_1}) \oplus \mathcal{O}_X(X_{f_2}) \cong A_{f_1} \times A_{f_2}$ is an isomorphism.

3. Let A be a commutative ring with 1, and M a maximal ideal, such that $X_1 := V(M)$ is an open (and closed) subset of $X := \mathrm{Spec}(A)$. Show, using Question 2, that $\mathcal{O}_X(X_1)$ is isomorphic to the local ring A_M. Conclude that A is isomorphic to $A_M \times \mathcal{O}_X(X_2)$, where $X_2 := X \setminus X_1$. Hint: See Homework 3 Question 9.

4. Mumford, Exercise following Example H in section II.1 on the arithmetic surface Spec$(\mathbb{Z}[X])$: Let $p \in \mathbb{Z}$ be a prime number, $f, g \in \mathbb{Z}[X]$ two irreducible elements of degree > 0 (i.e., each is an irreducible polynomial in $\mathbb{Q}[X]$, with integral coefficients, and the g.c.d of its coefficients is 1). Assume that $g \neq f$ and $g \neq -f$.

   (a) Describe all the points of $V((p)) \cap V((f))$.

   (b) Describe $Y := \mathrm{Spec}(\mathbb{Z}[X]/(p, f))$ and its structure sheaf. Hint: Show that each point $y \in Y$ is a connected component of Y. Conclude that $\{y\}$ is a distinguished open set Y_{e_y}, using Question 2. Describe $\mathcal{O}_Y(\{y\})$ in terms of p and f.

(c) Describe all the points of $V((f)) \cap V((g))$. Show, in particular, that it is a finite set and that all its points are closed points.

(d) Describe $\mathrm{Spec}(\mathbb{Z}[X]/(f,g))$ and its structure sheaf using Questions 2 and 3.

5. (Hartshorne, Exercise II.2.10) Describe $\mathrm{Spec}(\mathbb{R}[x])$. How does its topological space compare to the set $\mathbb{R}$? To $\mathbb{C}$?.

6. (a) (Hartshorne, Exercise II.2.8) Let X be a prescheme. For any point $x \in X$, let $\mathfrak{m}_x \subset \mathcal{O}_x$ be the maximal ideal of the stalk at x and $k(x) := \mathcal{O}_x/m_x$ the residue field. Define the *Zariski tangent space* T_xX to X at x to be the dual of the $k(x)$ vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a prescheme over a field k (see Definitions 1 and 2 in section II.3 of Mumford), and let $k[\epsilon]/(\epsilon^2)$ be the *ring of dual numbers* over k. Show that to give a k-morphism from Spec $[k[\epsilon]/(\epsilon^2)]$ to X is equivalent to giving a point $x \in X$, such that $k(x) = k$, and an element of T_xX.

   (b) Let k be an algebraically closed field, X an affine variety over k, $C \subset X$ a one dimensional closed subvariety, P a non-singular point of C, and $t \in \mathfrak{m}_{C,P} \subset \mathcal{O}_{C,P}$ a uniformizing parameter for the DVR $\mathcal{O}_{C,P}$. Construct a non-zero tangent vector in T_PX, which is tangent to C, in terms of the uniformizing parameter t.

7. (Hartshorne, Exercise 2.16) Let X be a prescheme, let $f \in \mathcal{O}_X(X)$, and define X_f to be the subset of points $x \in X$, such that the germ $f_x \in \mathcal{O}_x$ of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

   (a) Let $U = \mathrm{Spec}(B)$ be an open *affine* subscheme of X (see Question 1). Let $\bar{f} \in B = \mathcal{O}_U(U)$ be the restriction of f. Show that $U \cap X_f$ is equal to the distinguished open subset $U_{\bar{f}}$ of U. Conclude that X_f is an open subset of X.

   (b) Assume that X is quasi-compact. Set $A := \mathcal{O}_X(X)$ and let $a \in A$ be an element, whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. Hint: Use an open affine cover of X.

   (c) Now assume that X has a finite cover by open affines U_i, such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if the topological space of X is noetherian.) Let $b \in \mathcal{O}_{X_f}(X_f)$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

   (d) With the hypothesis of 7c conclude, that $\mathcal{O}_{X_f}(X_f) \cong A_f$.

8. (Hartshorne, Exercise 2.17, A *Criterion for Affineness*)

   (a) Let $f : X \to Y$ be a morphism of preschemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

   (b) Let X be a prescheme and set $A := \mathcal{O}_X(X)$. The prescheme X is an affine scheme, if and only if there is a finite set of elements $f_1, \ldots, f_r \in A$, such that the open subsets X_{f_i} are affine, and the ideal generated by $f_1, \ldots, f_r$ is the whole of A. Hint: Use Theorem 1 in section II.2 of Mumford, and excercise 7d above.

9. (Mumford, Problem after Definition 5 in section II.2) Let R be a commutative ring with a unit. Let $F \in R[X_0, \ldots, X_n]$ be a homogeneous polynomial of degree $d > 0$. Show that $\{P \in \mathbb{P}^n_R \ : \ F(P) \neq 0\}$ is a Zariski open subset of $\mathbb{P}^n_R$, which is isomorphic to

$$Spec\left(\frac{M_0}{F}, \ldots, \frac{M_N}{F}\right),$$

where M_i, $0 \leq i \leq N = \begin{pmatrix} n+d \\ d \end{pmatrix} - 1$ consists of all monomials $\prod_{i=0}^n X_i^{d_i}$ of degree d. Compare with Homework 4 Question 6.