On the growth of torsion in the cohomology of arithmetic groups

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Some data for our groups

Group	$\dim X$	vcd	cusp. range	δ
$GL_3(\mathbb{Z})$	5	3	$[2, 3]$	1
$GL_4(\mathbb{Z})$	9	6	$[4, 5]$	1
$GL_5(\mathbb{Z})$	14	10	$[6, 7, 8]$	2
$GL_2(\mathcal{O}_L)$	3	2	$[1, 2]$	1
$GL_2(\mathcal{O}_F)$	6	5	$[2, 3, 4]$	1
$GL_2(\mathcal{O}_E)$	7	6	$[2, 3, 4, 5]$	2

$L = \mathbb{Q}(\sqrt{-d})$, F cubic of discriminant -23, $E = \mathbb{Q}(\zeta_5)$. In all cases we use $\Gamma_0(\mathfrak{n})$ for our congruence subgroups.
Let c be the B–V constant $c_G \mu(\Gamma)$.

$\mathbb{Q}(\sqrt{-1})$

- $\Gamma \subset \mathrm{SL}_2(\mathcal{O})$, $X = \mathfrak{H}_3$
- $c = \frac{|\Delta|^{3/2}}{48\pi^3} \zeta_{\mathbb{Q}(\sqrt{-1})}(2) = 0.0080989140008\ldots$
- Computations done for $\mathrm{Norm}(\mathfrak{n}) \leq 50000$ (19827 levels).
- Largest torsion at norm 49850, where Voronoi homology is $H_1 = \mathbb{Z}^{18} \times T$,

$$
\begin{aligned}
\#T = &99407444600099014483472905584891296877204680639\\
&86416658793798948901127432947695155728875563424\\
&19476442159847189542963526150932346235466883619\\
&33161406412057509780714570218204049314881664033\\
&94721755271280981860183356597634324144243233944\\
&28888397376030584576028245868131438925540733906\\
&14865670538078059046800867779047996070659056392\\
&69615372231493648172254559736578451714510684160\\
&00000000000.
\end{aligned}
$$

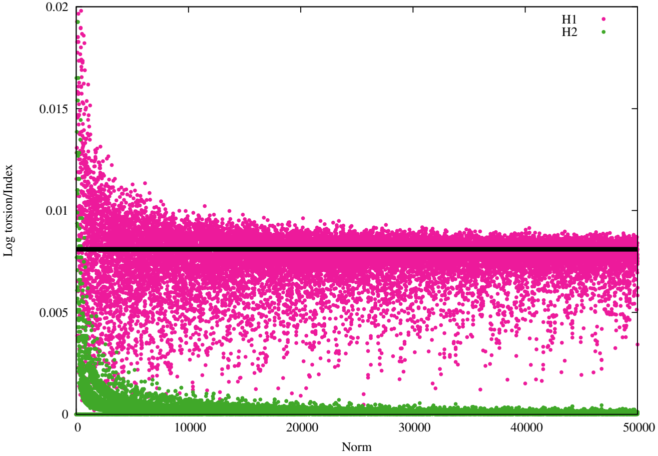


Figure: Levels ordered by norm

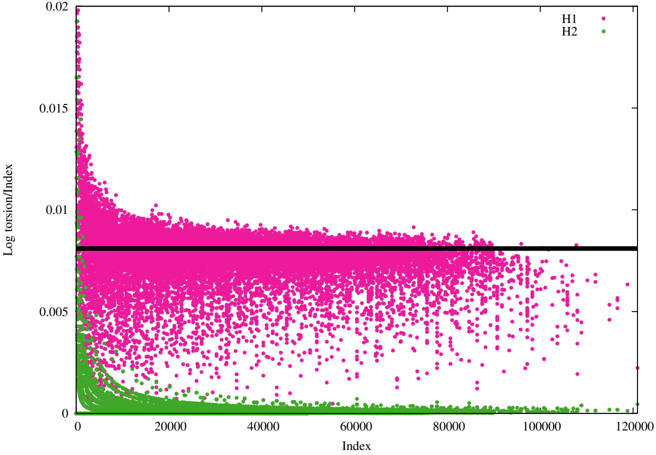


Figure: Levels ordered by index

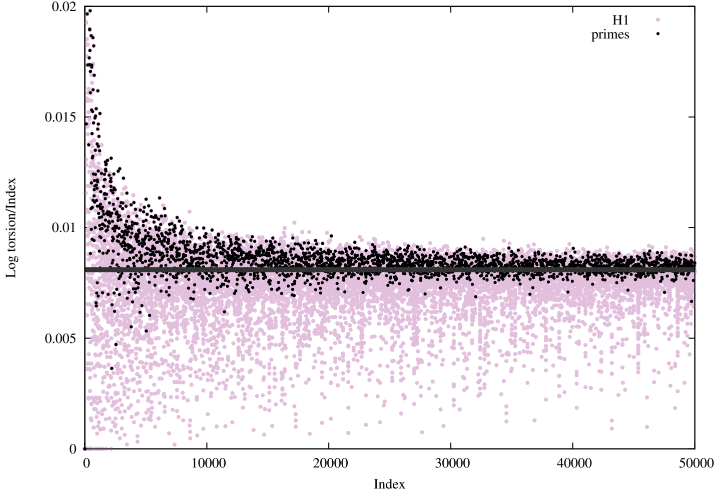


Figure: H_1 with prime levels indicated for the subgroups of $\mathrm{GL}_2(\mathbb{Z}[\sqrt{-1}])$.

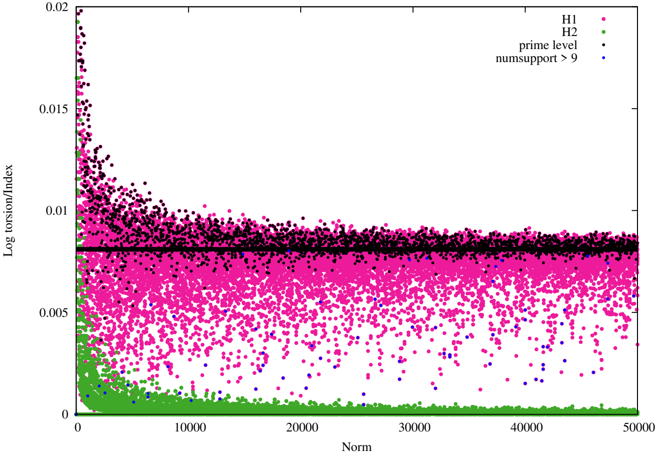


Figure: Levels ordered by norm, primes, bigsupport too

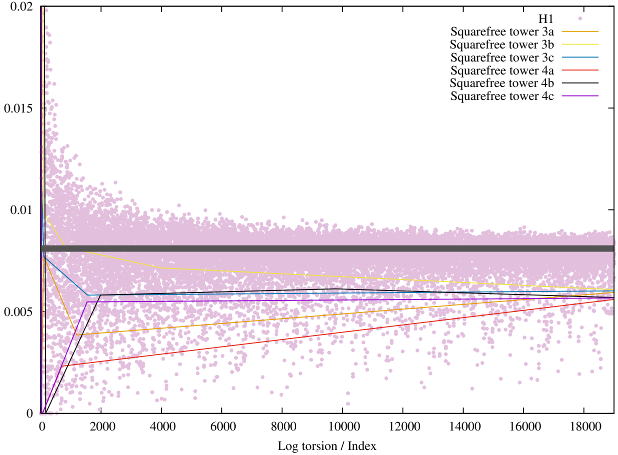


Figure: Some partial towers

$\mathbb{Q}(\sqrt{-15})$

- Class number $= 2$.
- $c = 0.0832543192934909\ldots$
- Computations done for $\mathrm{Norm}(\mathfrak{n}) \leq 10103$ (8303 levels).
- Largest torsion at norm 10020, where Voronoi homology is $H_1 = \mathbb{Z}^{142} \times T$,

$$
\begin{aligned}
\#T = & 41881066680290026290757971072933010839127589372 \\
& 20329609346932099823555080316242246455414143824 \\
& 81678312213487455195384194363167308898872657519 \\
& 11158997541503207392603276379894341069429480519 \\
& 65384392910119014805697326867603260287168237074 \\
& 47678067481735850787089416159137540458099351433 \\
& \quad \ldots 12 \text{ lines cut} \ldots \\
& 47612937080789420193496465314969566666312118346 \\
& 35590810694991462262604042802662380942618952274 \\
& 82502950783747405436363250199487566317958928712 \\
& 61212179944122598433961964350333355621142142537 \\
& 01636462958029126592626688000000000000000000000 \\
& 00000000000000000000000000000000000000000000000.
\end{aligned}
$$

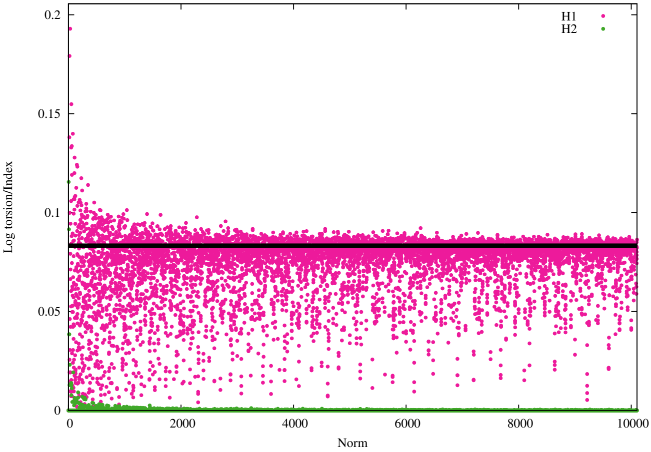


Figure: Levels ordered by norm

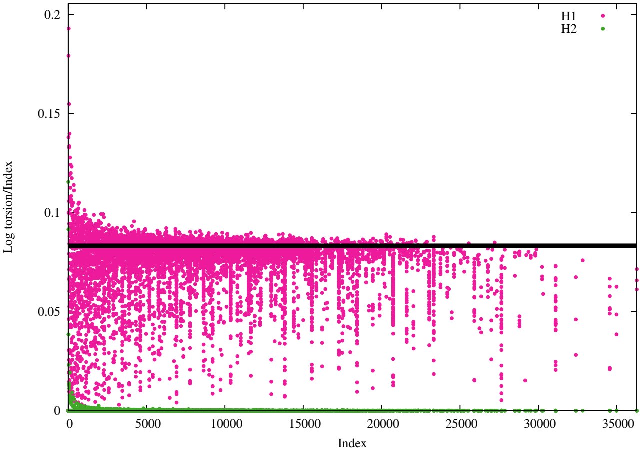
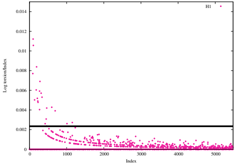


Figure: Levels ordered by index

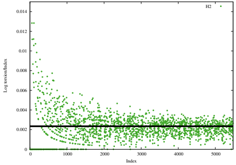
F = cubic field of discriminant -23

- $\Gamma \subset \mathrm{GL}_2(\mathcal{O}_F), X \simeq \mathfrak{H} \times \mathfrak{H}_3 \times \mathbb{R}$.
- $c = \frac{23^{3/2}\,\mathrm{Reg}_F}{48\pi^5}\zeta_F(2) = 0.002343900569\ldots$
- Full computations done for $\mathrm{Norm}(\mathfrak{n}) \leq 5480$ (2011 levels). We went further for the most interesting cohomology group.

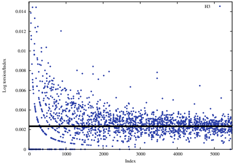
Note that the constant includes a factor for the regulator, since the symmetric space for GL_2 includes a flat factor (the locally symmetric space is an S^1-bundle over the SL_2 symmetric space).



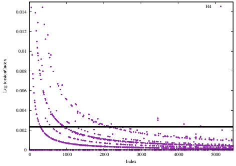
(a) H_1



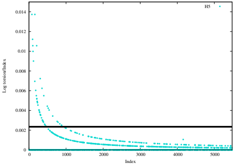
(b) H_2



(c) H_3



(d) H_4



(e) H_5

Figure: All the Voronoi homology groups for subgroups of $\mathrm{GL}_2(\mathcal{O}_F)$ for the cubic field of discriminant -23, together with the predicted limiting constant (ordered by index of the congruence subgroup). Cuspidal range is H_2, H_3, H_4.

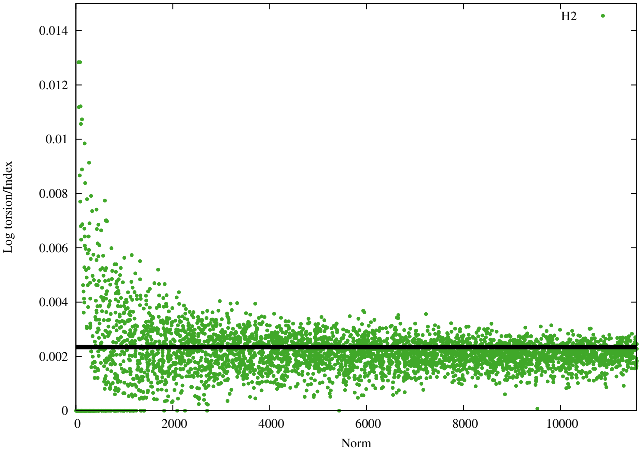


Figure: H_2, the most interesting group, by norm

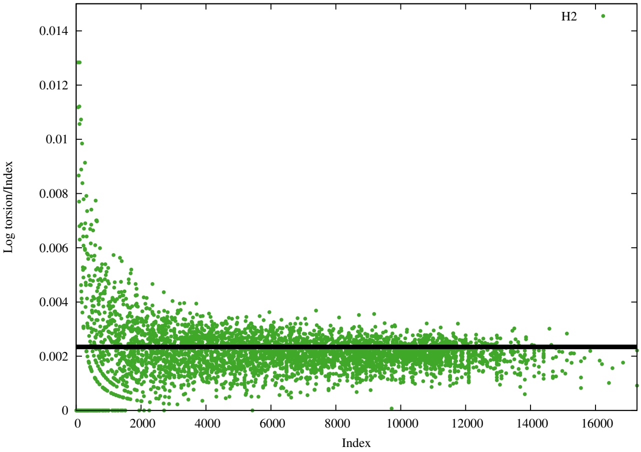


Figure: H_2, the most interesting group, by index

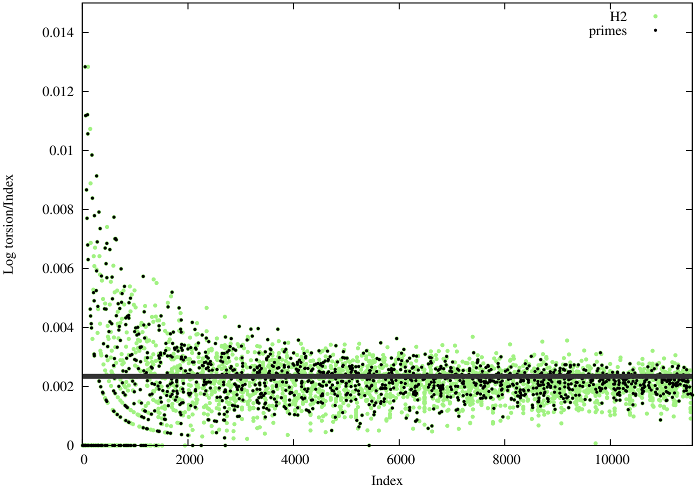


Figure: H_2 with prime levels indicated for the subgroups of $\mathrm{GL}_2(\mathcal{O}_F)$ for the cubic field of discriminant -23.

$E = \mathbb{Q}(\zeta_5)$

- $\Gamma \subset \mathrm{GL}_2(\mathcal{O}_E), X \simeq \mathfrak{H}_3 \times \mathfrak{H}_3 \times \mathbb{R}$.
- Don't expect exponential torsion growth ($\delta = 2$), so constant is 0.
- Computations done for Norm($\mathfrak{n}$) < 38172 (2741 levels)

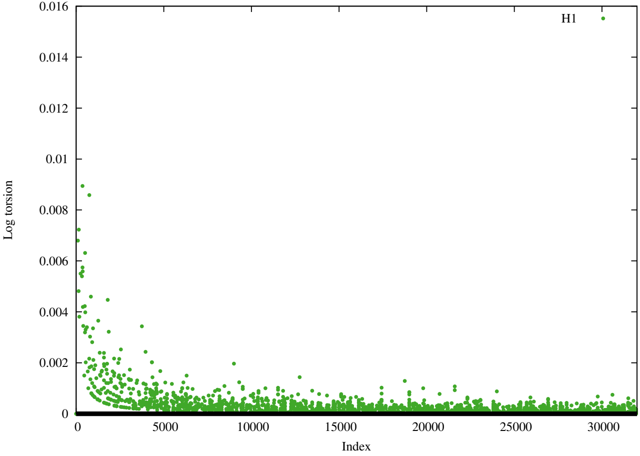
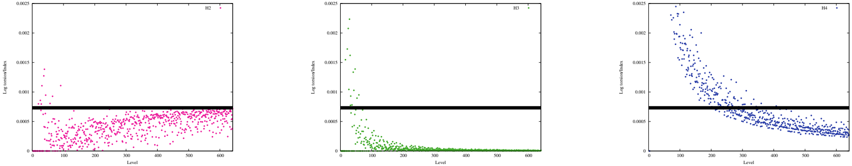


Figure: The (most interesting) group, ordered by index

$\mathrm{GL}_3 / \mathbb{Q}$

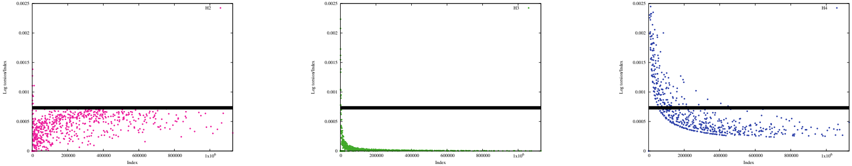
- $\Gamma \subset \mathrm{GL}_3(\mathbb{Z})$, X has dimension 5.
- $c = \frac{\sqrt{3}}{288\pi^2}\zeta(3) = 0.0007324760366280048 1\ldots$
- Computations (H_2) done for $\Gamma_0(N)$, $N \leq 641$.
- Largest torsion at $N = 570$, where Voronoi homology is $H_2 = \mathbb{Z}^{484} \times T$,

$$
\begin{aligned}
\#T = & 168532564282129269190913865060460075763037552 08 \\
& 26880272462076049132232810484870574950214286105 \\
& 73825604977439626031552020132671158394472458554 \\
& 36085727860222780889528730541550989755676579381 \\
& 17768448895558775766757399005134162840461734061 \\
& 64566680386962872504267631909519596190869144605 \\
& 43921348096819200.
\end{aligned}
$$



(a) H_2 (b) H_3 (c) H_4

Figure: All the Voronoi homology groups for subgroups of $\mathrm{GL}_3(\mathbb{Z})$, together with the predicted limiting constant (ordered by level). Cuspidal range: H_2, H_3.

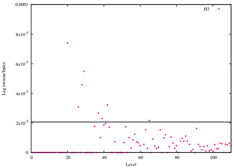


(a) H_2 (b) H_3 (c) H_4

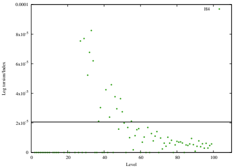
Figure: All the Voronoi homology groups for subgroups of $GL_3(\mathbb{Z})$, together with the predicted limiting constant (ordered by index). Cuspidal range: H_2, H_3.

$\mathrm{GL}_4/\mathbb{Q}$

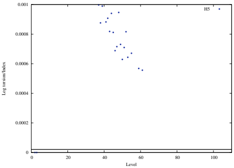
- $\Gamma \subset \mathrm{GL}_4(\mathbb{Z})$, X has dimension 9.
- $c = \frac{31\sqrt{2}}{259200\pi^2}\zeta(3) = 0.0000205999884056288780742643411677\ldots$
- Computations (H_3) done for $\Gamma_0(N)$, $N \leq 119$.
- Largest torsion at $N = 114$, where Voronoi homology is $H_3 = \mathbb{Z}^{69} \times T$,
$$\#T = 2^{12} \cdot 3^7 \cdot 11^4.$$



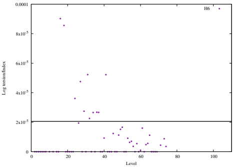
(a) H_3



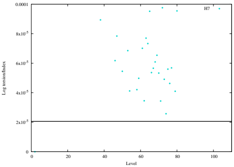
(b) H_4



(c) H_5

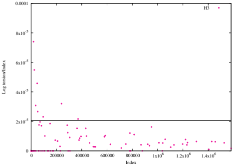


(d) H_6

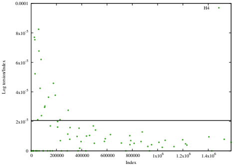


(e) H_7

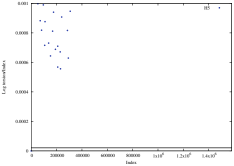
Figure: All the Voronoi homology groups for subgroups of $\mathrm{GL}_4(\mathbb{Z})$, together with the predicted limiting constant (ordered by level). Cuspidal range H_4, H_5.



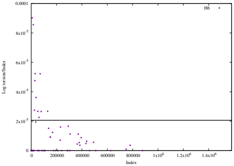
(a) H_3



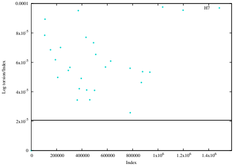
(b) H_4



(c) H_5



(d) H_6



(e) H_7

Figure: All the Voronoi homology groups for subgroups of $GL_4(\mathbb{Z})$, together with the predicted limiting constant (ordered by index of the congruence subgroup). Cuspidal range H_4, H_5.

Two conjectures

Conjecture. Let Γ be any arithmetic group. The limit

$$\lim_{k\to\infty} \frac{\log |H^i(\Gamma_k;\mathbb{Z})_{\mathrm{tors}}|}{[\Gamma:\Gamma_k]} \tag{1}$$

should tend to the B-V limit when $\delta = 1$ and when i is at the top of the cuspidal range and Γ_k ranges over congruence subgroups of Γ of *increasing prime level*.

Conjecture. Let Γ be any arithmetic group. The limit (1) should equal the B-V limit as long as Γ_k ranges over any set of congruence subgroups of *increasing level*. In particular, the lim inf

$$\liminf_{\Gamma_k} \frac{\log |H^i(\Gamma_k;\mathbb{Z})_{\mathrm{tors}}|}{[\Gamma:\Gamma_k]},$$

taken over all congruence subgroups, should equal the B-V limit.

Eisenstein cohomology and torsion

- Introduced by Harder.
- X^{BS} Borel–Serre compactification of X.
- $Y := \Gamma\backslash X$, $Y^{\mathrm{BS}} := \Gamma\backslash X^{\mathrm{BS}}$. We have

$$H^*(\Gamma; \mathbb{C}) \simeq H^*(Y; \mathbb{C}) \simeq H^*(Y^{\mathrm{BS}}; \mathbb{C}).$$

Eisenstein cohomology and torsion

- $\iota\colon \partial Y^{\mathrm{BS}} \to Y^{\mathrm{BS}}$, inclusion of the boundary.
- Interior cohomology $H_!^*(Y^{\mathrm{BS}};\mathbb{C})$ is the kernel of ι^*.
- The goal of Eisenstein cohomology is to construct a Hecke-equivariant section $s\colon H^*(\partial Y^{\mathrm{BS}};\mathbb{C}) \to H^*(Y^{\mathrm{BS}};\mathbb{C})$ mapping onto a complement $H^*_{\mathrm{Eis}}(Y^{\mathrm{BS}};\mathbb{C})$ of the interior cohomology.

Q: Do we see torsion Eisenstein classes? Of course any such classes would have to be constructed by topological means, not using Eisenstein series. So really we are asking if we see classes on locally symmetric spaces at infinity appearing in the cohomology of our locally symmetric space.

Eisenstein phenomena

We see apparent Eisenstein classes going from H_2 of GL_3 to H_3 of GL_4 (H_2 of GL_3 refers to H^3, and H_3 of GL_4 refers to H^6; these are the vcds).

- At level 114, the size of the torsion in H_3 is $2^{12} \cdot 3^7 \cdot 11^4$. The corresponding torsion for GL_3 in H_2 is $2^5 \cdot 3^3 \cdot 11^2$.
- At level 118, the size of the torsion in H_3 is $2^{14} \cdot 17^4$. The corresponding torsion for GL_3 in H_2 is 17^2.
- At level 119, the size of the torsion in H_3 is $2^4 \cdot 3^3 \cdot 31^4$. The corresponding torsion for GL_3 in H_2 is $2^2 \cdot 3^1 \cdot 31^2$.

Eisenstein phenomena

We also apparent Eisenstein classes for H_3 of GL_3 to H_4 for GL_4; both of these correspond to cohomological degree one below the vcd of their respective groups.

- At level 49, the size of the torsion in H_4 is $3^1 \cdot 7^2$. The corresponding torsion for GL_3 in H_3 is 7.
- At level 98, the size of the torsion in H_4 is 7^5. The corresponding torsion for GL_3 in H_3 is 7.

Summary

- We found excellent agreement in our results with the general heuristic espoused by Bergeron–Venkatesh, namely that groups with deficiency 1 should have exponential growth in the torsion in their cohomology. We also found excellent quantitative agreement with their predicted asymptotic limit, suitably interpreted for reductive groups.
- We found that, when the $\mathbb{Q}$-rank of a group is > 0 and the deficiency is 1, the explosive torsion growth occurs in the top cohomological degree of the cuspidal range and not in other degrees (after accounting for flat factors).
- When the deficiency is > 1, we still found interesting torsion in the top degree of the cuspidal range. However, the growth rate of the size of the torsion subgroup appears much lower than that in the deficiency 1 case. Is the growth polynomial or subexponential?

Summary

- For groups of deficiency 1, the growth of the torsion in towers of congruence subgroups seems to agree with the predicted asymptotic limit, although the convergence seems significantly slower than that experienced by families of congruence subgroups of increasing prime level or simply the family of all congruence subgroups ordered by increasing level.
- The interesting torsion in a group of deficiency 1 appears to tend to transfer to another via Eisenstein cohomology. What is the explanation of when this transfer happens and when it doesn't? Could this be related to divisibility of special values of some L-function by the primes in question?

Thanks

Thank you!