Toric varieties, modular forms, and L-functions

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Results

Let $\ell \geq 1$, $\Gamma_1(\ell) \subset SL_2(\mathbb{Z})$ be the congruence subgroup of matrices that are upper-triangular and unipotent mod ℓ, and let $\mathscr{M}_*(\ell) = \mathscr{M}_*(\Gamma_1(\ell); \mathbb{C})$ be the graded ring of holomorphic modular forms on $\Gamma_1(\ell)$. For weight 2, let $\mathscr{M}_2(\ell) = \mathscr{S}_2(\ell) \oplus \mathscr{E}_2(\ell)$ be the splitting into cusp forms and Eisenstein series

Using the geometry of compact toric varieties we construct a subring $\mathscr{T}_*(\ell) \subset \mathscr{M}_*(\ell)$. We show that $\mathscr{T}_*(\ell)$ is a natural subring, in that it's closed under many classical operations on modular forms:

- Hecke operators
- Fricke involution
- Atkin-Lehner lifting

Moreover, we can characterize $\mathscr{T}_2(\ell)$:

$$\begin{aligned}\mathscr{T}_2(\ell) = \quad & \{\mathbb{C}\text{-span of cuspidal eigenforms } f \\ & \text{with } L(1,f) \neq 0 \text{ (or lifts of such)}\} \\ & \oplus\{\text{some Eisenstein series}\}.\end{aligned}$$

Remarks.

1. Principal motivation was Borisov-Libgober's computation of the Witten genus for complete toric varieties with mild singularities. This produces modular forms on $\Gamma_0(2)$.

2. $\mathscr{T}_*(\ell)$ is related to Hirzebruch elliptic genera of level ℓ.

3. The construction of $\mathscr{T}_*(\ell)$ is elementary from a number-theoretic point of view. In particular, $\mathscr{T}_2(\ell)$ is constructed without using heavy machinery like the Shimura correspondence. Theta functions do appear (in particular ϑ_{11}), but no quaternion algebras, Brandt matrices, etc.

Toric varieties and fans

Let $N \simeq \mathbb{Z}^d$ be a lattice, and let $\hat{N} = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. We denote the pairing by a dot: $(n, \hat{n}) \mapsto n \cdot \hat{n}$. Let $N_{\mathbb{R}} = N \otimes \mathbb{R}$.

Definition. $C \subset N_{\mathbb{R}}$ is a *rational polyhedral cone* if C is the convex hull of finitely many N-rational rays, and contains no line. The *dual cone* $\hat{C} \subset \hat{N}_{\mathbb{R}}$ is the set of linear forms nonnegative on all of C.

Definition. A *complete fan* Σ is a set of rational polyhedral cones satisfying:

- If $C \in \Sigma$, then any face of C is in Σ.
- If $C, C' \in \Sigma$, then $C \cap C'$ is a face of each.
- $\bigcup_{C \in \Sigma} C = N_{\mathbb{R}}$.

Using a fan Σ we can construct a toric variety X_Σ:

$$\begin{array}{ccc} \{\text{Cones}\} & \Longrightarrow & \{\text{f.g. } \mathbb{C}\text{-algebras}\}, \\ C & \longmapsto & \mathbb{C}[\hat{C} \cap \hat{N}], \end{array}$$

and we set $U_C = \operatorname{Spec} \mathbb{C}[\hat{C} \cap \hat{N}]$. If $C'' \hookleftarrow C \hookrightarrow C'$ then $U_{C''} \hookleftarrow U_C \hookrightarrow U_{C'}$. So the combinatorics of Σ tells us how to glue together the U_C's to get a complete variety X_Σ.

X_Σ has a $T = (\mathbb{C}^\times)^d$ action, and 1-cones correspond to T-stable divisors. If X_Σ is nonsingular, then the classes of these divisors generate the cohomology ring.

Example.

- (Complete fans) $\mathbb{P}^n$, $\prod \mathbb{P}^{n_i}$, . . .
- (Incomplete fans) $\mathbb{C}^n$, $\operatorname{Spec}(\mathbb{C}[x,y,z]/(xy - z^2))$, . . .

Degree functions and modular forms

Definition. A function $\deg\colon N \to \mathbb{C}$ is a *degree function* if it's piecewise-linear and linear on the cones of some fan Σ.

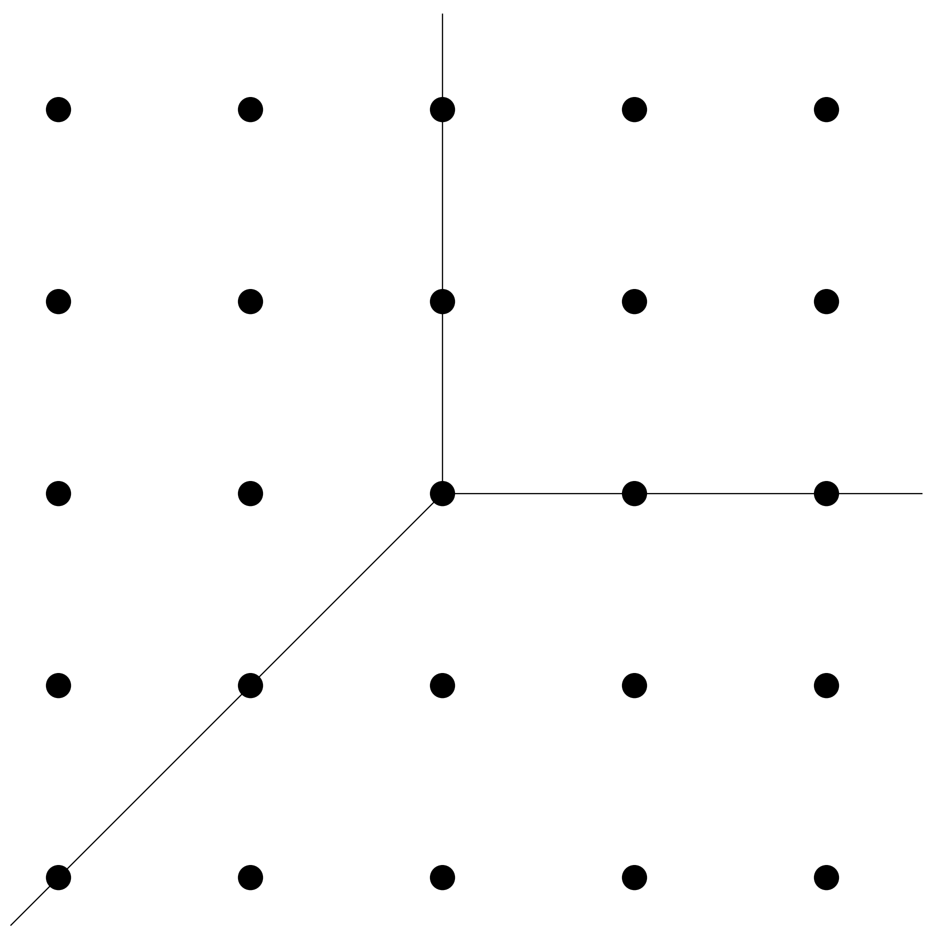
Definition. Let $\deg$ be a degree function. We define $f = f_{N,\deg}\colon \mathfrak{H} \to \mathbb{C}$ by

$$f(\tau) := \sum_{\hat{n}\in\hat{N}} \sum_{C\in\Sigma} (-1)^{\operatorname{codim} C} \sum_{n\in C\cap N} q^{n\cdot\hat{n}} \mathrm{e}^{2\pi \mathrm{i} \deg(n)}.$$

Theorem. [B–G] Let $\ell > 1$ be an integer, and let $\deg\colon N \to (1/\ell)\mathbb{Z}$ be a degree function with respect to a fan Σ. Suppose $\deg(x) \notin \mathbb{Z}$ if x is the primitive generator of any 1-cone of Σ. Then $f_{N,\deg} \in \mathscr{M}_d(\ell)$.

Example. $X = \mathbb{P}^2$, $\deg(x) = 1/2$ for all primitive generators of 1-cones. Then

$$f = \sum_{a,b \in \mathbb{Z}} \frac{2}{(1+q^a)(1+q^b)(1+q^{-a-b})} \in \mathscr{M}_2(2).$$



To prove the theorem there are several steps:

- First we must show that $f_{N,\deg}$ is well-defined as a power series in q, i.e. that only finitely many terms will contribute to a given power of q. This can be done by reinterpreting f in terms of a construction in homological algebra.

- Next we need to show modularity. Rather than deal with the definition of f directly, we use a different expression that's more convenient. We construct a certain infinite-dimensional q-graded vector bundle W over X. By Hirzebruch-Riemann-Roch, we know that $\chi(W) = \int_X ch(W)Td(X)$. The right-hand side is computed to be

$$\int_X \prod_i \frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i} - \alpha_i, \tau)\vartheta'(0,\tau)}{\vartheta(D_i/2\pi\mathrm{i}, \tau)\vartheta(-\alpha_i, \tau)}$$

where the D_i are the classes of the T-stable divisors in the cohomology ring of X, and where

$$\vartheta(z,\tau) = \vartheta_{11}(z,\tau) = \frac{1}{i}\sum_{n\in\mathbb{Z}}(-1)^n \mathrm{e}^{\pi \mathrm{i}\tau(n+\frac{1}{2})^2}\mathrm{e}^{\pi \mathrm{i} z(2n+1)}$$

is the theta function with characteristic $(1/2, 1/2)$. The expression in the integral is interpreted using the Jacobi triple product formula for ϑ.

The left-hand side can be computed using Cech cohomology, and we get the original expression for f.

- We show boundedness at the cusps later, after we identify some distinguished forms in $\mathscr{T}_*(\ell)$.

Hecke action

Hecke operators act on forms in $\mathscr{T}_*(\ell)$ by taking sublattices of the lattice N. For example, let p be a prime with $(p, \ell) = 1$, and let $f \in \mathscr{T}_d(\ell)$. Then

$$f_{N,\deg} \mid T_p = \sum_S f_{S,p\deg} + \frac{p - p^{d-1}}{p-1} f_{N,\deg},$$

where the sum is taken over lattices S satisfying $N \subset S \subset \frac{1}{p}N$ and $[S : N] = p^{d-1}$.

Similar expressions can be obtained for the operators U_p when $(p, \ell) > 1$, and the Fricke involution.

Generators for $\mathscr{T}_*(\ell)$

The multiplication

$$f_{N,\deg} f_{N',\deg'} = f_{N\oplus N',\deg\oplus\deg'}$$

puts a ring structure on $\mathscr{T}_*(\ell)$, and we have the following:

Theorem. [B–G] Let $\ell > 5$. Then $\mathscr{T}_*(\ell)$ is generated as a graded ring by the weight one Eisenstein series $s_a(q)$, where $a = 1, \ldots, \ell - 1$, and

$$s_a(q) = \frac{\mathrm{e}^{2\pi\mathrm{i}a/\ell} + 1}{2(\mathrm{e}^{2\pi\mathrm{i}a/\ell} - 1)} - \sum_d q^d \sum_{k|d} (\mathrm{e}^{2\pi\mathrm{i}ka/\ell} - \mathrm{e}^{-2\pi\mathrm{i}ka/\ell}).$$

Note that s_a only depends on a modulo ℓ.

For $\ell \leq 5$ the result is mildly more complicated. The Eisenstein series s_a comes from the toric variety $\mathbb{P}^1$: simply put $\deg(1) = \deg(-1) = a/\ell$ in the unique complete fan in $\mathbb{R}$.

Nonvanishing of L-functions

We want to prove that $\mathscr{T}_2(\ell) \cap \mathscr{S}_2(\ell)$ consists of exactly the span of the cuspidal eigenforms with nonvanishing L-function at the center of the critical strip. To do this we use Manin symbols.

Let $\ell > 1$ be an integer, and let $E_\ell \subset (\mathbb{Z}/\ell\mathbb{Z})^2$ be the subset of pairs (u, v) such that $\mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}/\ell\mathbb{Z}$. The space of *Manin symbols* M is the $\mathbb{C}$-vector space generated by the symbols $(u, v) \in E_\ell$ modulo the relations:

1. $(u, v) + (-v, u) = 0$.

2. $(u, v) + (v, -u - v) + (-u - v, v) = 0$.

We have subspaces $M_\pm \subset M$ corresponding to the eigenspaces of the involution $(u, v) \mapsto (-u, v)$, and we have symmetrization maps $(\ ,\ )_\pm\colon M \to M_\pm$ given by $(u, v)_\pm := ((u, v) \pm (-u, v))/2$.

The pairing

Manin symbols correspond to paths on the modular curve $X_1(\ell)$. Given (u, v), choose cusps $\alpha = a/u$ and $\beta = b/v$ such that $av - bu = 1$. Then we get a path on $X_1(\ell)$ by projecting the geodesic between α and β on $\mathfrak{H}^*$.

We get a pairing

$$M \times \mathscr{S}_2(\ell) \longrightarrow \mathbb{C}$$

by integration. This pairing is nondegenerate when restricted to the plus (resp. minus) *cuspidal Manin symbols* S_+ (resp. S_-). These are the symbols that induce cycles on the modular curve, not just cycles mod the cusps.

Note that integration gives a map $\mathscr{S}_2(\ell) \to M_+^*$.

From M_- to $\mathscr{S}_2(\ell)$ via $\mathscr{T}_2(\ell)$

The Eisenstein series s_a satisfy certain relations:

- $s_{-a} = -s_a$
- If $a + b + c = 0 \bmod \ell$, then $s_a s_b + s_b s_c + s_c s_a$ is an Eisenstein series.

So we can define a map

$$\mu\colon M_- \longrightarrow \mathscr{M}_2(\ell)/\mathscr{E}_2(\ell) \simeq \mathscr{S}_2(\ell)$$

by $(a, b) \mapsto s_a s_b \bmod \mathscr{E}_2(\ell)$. (Actually we compose this with the Fricke involution, but this is not important for this talk.)

A projection map

To get the nonvanishing result, we construct a linear operator

$$\rho \colon \mathscr{S}_2(\ell) \longrightarrow \mathscr{S}_2(\ell)$$

that kills cuspforms with $L(1, f) = 0$. Then we show that μ factors ρ.

The definition of ρ is

$$\rho(f) = \sum_{n=1}^{\infty} \left(\int_0^{\mathrm{i}\infty} (f \,|\, T_n)(s) ds \right) q^n,$$

where T_n is the nth Hecke operator. By a result of Merel, this can be written as

$$-\sum_{n>0} q^n \sum_{ad-bc=n, a>b\geq 0, d>c\geq 0} \varphi((c,d)_+),$$

where $\varphi \in M_+^*$ corresponds to f.

Factoring ρ

The final ingredient is a map $\pi\colon M_+^* \to M_-$ constructed using the intersection pairing. We omit the definition (a similar map was recently studied by Merel). The result is the following:

Theorem. The composition of the maps in the diagram

$$\mathscr{S}(l) \xrightarrow{\int} M_+^* \xrightarrow{\pi} M_- \xrightarrow{\mu} \mathscr{M}(l)/\mathscr{E}(l) \xrightarrow{\sim} \mathscr{S}(l)$$

is -12ρ.

We prove this theorem by explicitly computing with the Fourier expansions of s_a's, and comparing it to the expression for ρ in terms of the Manin symbols.

In progress

- We're studying $\mathscr{T}_*(\ell)$ in higher weight. Our conjecture is that modulo Eisenstein series $\mathscr{T}_*(\ell) = \mathscr{M}_*(\ell)$ for sufficiently high weight (in fact for weights ≥ 3).

- In joint work with Sorin Popescu, we're studying (for ℓ prime) the map

  $$X_1(\ell) \longrightarrow \mathbb{P}(\mathbb{C}[s_a \mid a = 1, \ldots, \ell - 1]).$$

  We hope the geometry of this map can give a bound on the size of $\mathscr{T}_2(\ell)$.