Computing Hecke operators on Siegel modular forms

PAUL E. GUNNELLS
(joint work with Mathieu Dutour Sikirić)

Let $G = \mathrm{Sp}_4(\mathbb{R})$ be the Lie group of 4×4 symplectic matrices and let $K = \mathrm{U}(2)$ be a maximal compact subgroup. The symmetric space $\mathfrak{H}_2 = G/K$ can be identified with the Siegel upper halfspace of degree 2 (the space of 2×2 symmetric complex matrices with positive-definite imaginary part). Let $\Gamma \subset \mathrm{Sp}_4(\mathbb{Z})$ be a level N congruence subgroup. The locally symmetric space $\Gamma\backslash\mathfrak{H}_2$ is a Siegel modular threefold, and is a moduli space of abelian surfaces with level structure related to Γ.

Our main goal is explicitly computing the cohomology spaces $H^*(\Gamma\backslash\mathfrak{H}_2, \mathbb{C})$, or more generally $H^*(\Gamma\backslash\mathfrak{H}_2, \mathcal{M})$, where $\mathcal{M}$ ranges over certain complex local systems on the threefold. We are especially interested in $H^3(\Gamma\backslash\mathfrak{H}_2, \mathcal{M})$, which is known to be computable in terms of certain (vector-valued) Siegel modular forms. Furthermore, we want to understand H^3 not just as a vector space, but as a Hecke module. More precisely, for each prime $p \nmid N$ there are two Hecke operators $T_{p,1}, T_{p,2}$ acting on the cohomology, and we want to understand the decomposition into eigenspaces. Such computations are essential to understand the arithmetic nature of the cohomology. Our eventual application will be to test conjectures of Harder, which uses the critical values of certain L-functions to predict congruences between vector-valued Siegel modular forms and elliptic modular forms. For more details about Siegel modular forms, their relations with cohomology, the Hecke operators, and Harder's conjectures, we refer to [8, 13].

Before describing our techniques, we give a selected overview of prior related work. Poor–Yuen [12], in their computational investigation of Brumer–Kramer's paramodular conjecture—which predicts that the L-functions of certain abelian surfaces should agree with the spinor L-functions of certain Siegel modular forms of paramodular type—computed weight 2 and 3 Siegel paramodular forms of prime levels < 600 along with the Hecke operators. We remark that the weight 2 forms are not cohomological, in that they cannot appear in the cohomology of the Siegel modular variety. Their techniques use theta series and the product structure on Siegel modular forms in an essential way. Cunningham–Dembélé [3] use the technique of algebraic modular forms. In particular they take a real quadratic field F, a quaternion algebra B/F ramified only at the two real places, and then use the Jacquet–Langlands correspondence to pass from automorphic forms on the group of unitary similitudes $\mathrm{GU}_2(B)$ to the forms on the group of symplectic similitudes $\mathrm{GSp}_4(F)$. Finally, Faber–van der Geer [5, 6] treated the case of full level ($N = 1$) by using the moduli space interpretation of $\mathrm{Sp}_4(\mathbb{Z})\backslash\mathfrak{H}_2$. In particular they made lists of $\mathbb{F}_q$-isomorphism classes of genus two curves with their automorphisms, and used this data to compute the traces of the Hecke operators on the cohomology of the Siegel modular threefold. This enabled them to provide convincing evidence for Harder's conjectures in many cases [13].

We now turn to our techniques. Our work uses tools similar to those found in modular symbols calculations [2, 11]. In particular, (i) we compute cohomology using an explicit finite cell complex that comes from considering an infinite cell complex with Γ-action; and (ii) the Hecke operators do not act on the cells of the complex, but we have an algorithm that allows us to write the Hecke image of any cycle in terms of cycles supported on the complex.

First we consider the complex. We use the reduction theory for $\mathrm{Sp}_4(\mathbb{R})$ due to McConnell–MacPherson [10]. This constructs a $\mathrm{Sp}_4(\mathbb{Z})$-equivariant cell decomposition of the symmetric space $\mathfrak{H}_2$ using Voronoi's explicit reduction for positive-definite real quadratic forms. The data indexing the cells, which are lists of primitive integral vectors in $\mathbb{Z}^4$, can be found in [9, 10]. We remark that the top-dimensional cells in this complex are *not* fundamental domains for the action of $\mathrm{Sp}_4(\mathbb{Z})$ on $\mathfrak{H}_2$, but are not far from it: the action of $\mathrm{Sp}_4(\mathbb{Z})$ on the cells has only finite stabilizers, and one can use the knowledge of the boundary maps and the stabilizers to compute the cohomology of $\Gamma\backslash\mathfrak{H}_2$ with coefficients in the local systems $\mathcal{M}$. Moreover, the stabilizer subgroups themselves can easily be computed from the data in [9, 10]. The picture the reader should keep in mind is the Farey tessellation in the elliptic modular case. The upper half plane $\mathfrak{H}_1$ can be $\Gamma' = \mathrm{SL}_2(\mathbb{Z})$-equivariantly tessellated by the Γ' orbit of the ideal triangle Δ with vertices at $0, 1, \infty$. The triangle Δ is not a fundamental domain for Γ', but a subdivision of Δ into three smaller triangles is.

Next we consider the Hecke operators. For simplicity we discuss how the algorithm computes on H^4, which is considerably simpler than H^3. We also discussing only the case of trivial coefficients, so that we can focus on the geometry of the problem. This is the direct analogue of the classical modular symbols case for $\mathrm{SL}_2(\mathbb{Z})$; our techniques now are already significantly different from the usual modular symbol algorithm (continued fractions), which was worked out for the symplectic group in [7].

Let $x_0 \in \mathfrak{H}_2$ be the basepoint corresponding to the maximal compact subgroup K, and let T be the standard maximal torus in $\mathrm{Sp}_4(\mathbb{R})$. The orbit $T \cdot x_0$ is a 2-dimensional subset in $\mathfrak{H}_2$, and by duality represents a cohomology class in $H^4(\Gamma\backslash\mathfrak{H}_2; \mathbb{C})$. Under a Hecke operator, the orbit $T \cdot x_0$ is taken to a finite set of orbits $\{T_i \cdot y_i\}$, where each T_i is now a rational conjugate of T and y_i is some other point in $\mathfrak{H}_2$. We must find a homology between each of these new subsets and cycles supported on $\mathrm{Sp}_4(\mathbb{Z})$-translates of $T \cdot x_0$, because it is these translates that form the 2-cells of our cell complex.

To do this, we work in a certain Satake (partial) compactification $\mathfrak{H}_2^*$ of $\mathfrak{H}_2$ [1]. This enlarges $\mathfrak{H}_2$ by adjoining copies of the upper half plane $\mathfrak{H}_1$ and points at infinity, much in the same way that the upper half plane is enlarged by adding cusps. Indeed, the construction is hereditary, in that the single points we add to $\mathfrak{H}_2$ are actually the cusps of the upper half planes we add at infinity. The cell decomposition of $\mathfrak{H}_2$ extends to $\mathfrak{H}_2^*$, and on the boundary components one sees the Farey tessellation. Let O be a Hecke image $T_i \cdot y_i$ and let $\bar{O}$ be its closure in $\mathfrak{H}_2^*$. Then the "edges" $\partial O := \bar{O} \smallsetminus O$ appear in certain boundary components as ideal geodesics going from cusp to cusp, and cutting across the edges of the Farey tessellation. As a first step in finding a cellular representative for the class of $\bar{O}$, we "fix" the edges of ∂O: we apply the classical modular symbol algorithm for $\mathrm{SL}_2(\mathbb{Z})$ to first write the boundary ∂O as a 1-cycle $\eta = \sum n_i \gamma_i$, where the γ_i are edges in the boundary tessellation.

Next we must fill in the 1-cycle: we must find a 2-chain ξ supported on the cells of our complex such that $\partial \xi = \eta$. Such a 2-chain is exactly our representative for the class of our Hecke image. To do this, we simply take a large set of top-dimensional cells $C_1, \dots, C_k$ that covers $\bar{O}$ and the support of η. We then attempt to solve the equation $(*)$ $\partial \xi = \eta$ with a 2-cycle supported on the 2-faces of the C_i.

Any such solution is exactly what we need, as it gives a representative for the class of our Hecke image supported on the complex. Moreover, we are guaranteed to succeed: *if we have sufficiently many C_i, we know that a solution exists.* Note that there is no complicated geometry needed to find ξ as in [7]; it is simply a problem in numerical linear algebra. We take many C_i and try to solve $(*)$; if we are unsuccessful, we add more top cells and try again. Eventually we will succeed.

We remark that for practical computations it is not enough to simply solve $(*)$. We need to find a solution to $(*)$ supported on as few 2-cells as possible. This can be done using a tool from applied mathematics, namely *compressed sensing*. Compressed sensing is a signal processing technique for acquiring and reconstructing a signal by finding solutions to underdetermined linear systems. The underlying problem of finding sparse solutions of linear systems is called *basis pursuit*; in our application we solve this problem using the approximate message passing algorithm proposed in [4].

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