Hecke operators, automorphic forms, and the cohomology of arithmetic groups: I

Paul E. Gunnells

UMass Amherst

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Main Point/Overview

Via the cohomology of arithmetic groups one can computationally investigate certain automorphic forms.

I: general picture + a higher $\mathbb{Q}$-rank example (PG)

II: some implementation details for $\mathbb{Q}$-rank 1 examples (DY)

In particular . . .

The cohomology of any arithmetic group is built out of certain automorphic forms, yet can be computed using topological tools.

- Gives a concrete way to compute automorphic forms that complements other approaches (e.g., theta series, explicit Jacquet–Langlands).
- Gives explicit examples of various constructions in automorphic forms (e.g., functorial liftings).
- Gives examples of automorphic forms that should be related to arithmetic objects (e.g., Galois representations). Enables testing various "motivic $\Longrightarrow$ automorphic" conjectures.

Geometric setup

$\mathbf{G}$	reductive connected algebraic group $/\mathbb{Q}$
$G = \mathbf{G}(\mathbb{R})$	group of real points (Lie group)
$K \subset G$	maximal compact subgroup
$A_G \subset G$	connected component of group of real points of maximal $\mathbb{Q}$-split torus in the center of G
$X = G/A_G K$	global symmetric space
$\Gamma \subset \mathbf{G}(\mathbb{Q})$	arithmetic subgroup
E	finite-dimensional rational complex representation of $\mathbf{G}(\mathbb{Q})$

Cohomology

The quotient $\Gamma\backslash X$ is a locally symmetric space (sometimes even an algebraic variety).

If Γ is torsion-free, then $\Gamma\backslash X$ is a manifold and even an Eilenberg–Mac Lane space ($\pi_1(\Gamma\backslash X) \simeq \Gamma$, higher homotopy groups vanish). We have

$$H^*(\Gamma; E) = H^*(\Gamma\backslash X; \mathscr{E}),$$

where $\mathscr{E}$ is the local coefficient system attached to E.

In fact, this isomorphism is true even if Γ has torsion, since we're using complex representations.

These are the cohomology spaces that realize automorphic forms.

Example: Classical modular forms

$\mathbf{G}$	$\mathrm{SL}_2/\mathbb{Q}$
G	$\mathrm{SL}_2(\mathbb{R})$
K	$\mathrm{SO}(2)$
A_G	trivial
X	the upper halfplane $\mathfrak{H}$
$\Gamma \subset \mathbf{G}(\mathbb{Q})$	congruence subgroup $\Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$
E	$M_{k-2}(\mathbb{C})$, degree $k-2$ homogeneous polynomials in $\mathbb{C}[x, y]$

Example: Classical modular forms

The quotient $\Gamma\backslash X$ is an open modular curve.

We have

$$H^1(\Gamma\backslash X; \mathscr{E}) \simeq S_k(\Gamma) \oplus \overline{S}_k(\Gamma) \oplus \mathrm{Eis}_k(\Gamma),$$

where S_k is the space of holomorphic weight k modular forms, and Eis_k is the space of weight k Eisenstein series.

Example: Biianchi modular forms

$\mathbf{G}$	$R_{F/\mathbb{Q}}\mathrm{SL}_2$, where $F/\mathbb{Q}$ imaginary quadratic
G	$\mathrm{SL}_2(\mathbb{C})$
K	$\mathrm{SU}(2)$
A_G	trivial
X	hyperbolic 3-space $\mathfrak{H}_3$
$\Gamma \subset \mathbf{G}(\mathbb{Q})$	congruence subgroup $\Gamma_0(\mathfrak{n}) \subset \mathrm{SL}_2(\mathscr{O})$
E	$M_{n,m}(\mathbb{C}) = M_n(\mathbb{C}) \otimes \overline{M}_m(\mathbb{C})$, where $m+n$ is even

Example: Biianchi modular forms

The quotient $\Gamma\backslash\mathfrak{H}_3$ is a hyperbolic 3-orbifold. Note that it has no complex structure.

The cohomology H^k vanishes if $k \geq 3$. The classes corresponding to cusp forms live in degrees $k = 1, 2$, and only occur for the modules $M_{n,n}(\mathbb{C})$.

Example: Hilbert modular forms

$\mathbf{G}$	$R_{F/\mathbb{Q}}\mathrm{GL}_2$, where F is totally real, degree d
G	$\mathrm{GL}_2(\mathbb{R}) \times \cdots \times \mathrm{GL}_2(\mathbb{R})$, d factors
K	$\mathrm{O}(2) \times \cdots \times \mathrm{O}(2)$, d factors
A_G	$\mathbb{R}_{>0}$
X	$\mathfrak{H} \times \cdots \times \mathfrak{H} \times \mathbb{R}^{d-1}$
$\Gamma \subset \mathbf{G}(\mathbb{Q})$	congruence subgroup $\Gamma_0(\mathfrak{n}) \subset \mathrm{GL}_2(\mathscr{O})$
E	essentially $\bigotimes_{i=1}^{d} M_{k_i}(\mathbb{C})$

Example: Hilbert modular forms

The quotient $\Gamma\backslash X$ is a torus bundle over a Hilbert modular variety.

This time the cohomology spaces H^k for $k = d, \ldots, 2d-1$ contain classes corresponding to cusp forms. (Note that $\dim_{\mathbb{R}}(\Gamma\backslash X) = 3d-1$.)

If we used SL_2 instead of GL_2 we'd get a Hilbert modular variety with real dimension $2d$, and the cusp forms would contribute to H^d.

$(\mathfrak{g}, K)$-cohomology

What's the connection between automorphic forms and cohomology?

The first step is the de Rham theorem. Let's assume **G** is semisimple.

Let $\Omega^p = \Omega^p(X, E)$ be the space of E-valued p-forms on X.

Let $\Omega^p(X, E)^\Gamma$ be the subspace of Γ-invariant forms.

We have a differential $d\colon \Omega^p \to \Omega^{p+1}$ and have an isomorphism

$$H^*(\Gamma; E) = H^*(\Omega^*(X, E)^\Gamma).$$

$(\mathfrak{g}, K)$-cohomology

We can identify

$$\Omega^p(\Gamma\backslash X, \mathbb{C}) = \mathrm{Hom}_K(\wedge^p(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G)),$$

or more generally

$$\Omega^p(\Gamma\backslash X, E) = \mathrm{Hom}_K(\wedge^p(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G) \otimes E).$$

Here K acts on the first entry by the adjoint representation and on the second entry by right translation.
RHS inherits a differential (or one can define it directly using relative Lie algebra cohomology). The cohomology of this complex is denoted

$$H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E)$$

and is called $(\mathfrak{g}, K)$-*cohomology*.

Cuspidal and square-integrable cohomology

Thus we have

$$H^*(\Gamma; E) = H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E).$$

We can use this to identify important subspaces of the cohomology, but we need more notation.

Consider the space $L^2(\Gamma\backslash G)$. We have the subspace

$$L^2_{\mathrm{disc}}(\Gamma\backslash G) \subseteq L^2(\Gamma\backslash G)$$

consisting of the direct sum of the irreducible subspaces. Inside this we have the G-invariant subspace of the *cusp forms*

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G) \subseteq L^2_{\mathrm{disc}}(\Gamma\backslash G) \subseteq L^2(\Gamma\backslash G).$$

Cuspidal and square-integrable cohomology

At the level of smooth vectors we have inclusions

$$L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty \subseteq L^2_{\mathrm{disc}}(\Gamma\backslash G)^\infty \subseteq C^\infty(\Gamma\backslash G).$$

Plug this into

$$H^*(\Gamma; E) = H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E).$$

and get an injective map

$$H^*(\mathfrak{g}, K; L^2_{\mathrm{cusp}}(\Gamma\backslash G)^\infty \otimes E) \to H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E).$$

The image $H^*_{\mathrm{cusp}}(\Gamma; E) \subset H^*(\Gamma; E)$ is called the *cuspidal cohomology*.

Borel conjecture

We also have the subspace of automorphic forms

$$A(\Gamma, G) \subset C^\infty(\Gamma \backslash G)$$

(subspace of functions that are right K-finite, left $Z(\mathfrak{g})$-finite, and of moderate growth).

Theorem (Franke)

The inclusion $A(\Gamma, G) \to C^\infty(\Gamma \backslash G)$ induces an isomorphism

$$H^*(\mathfrak{g}, K; A(\Gamma, G) \otimes E) \to H^*(\mathfrak{g}, K; C^\infty(\Gamma \backslash G) \otimes E).$$

Voilà

Thus we can think of $H^*(\Gamma; E)$ as being a concrete realization of certain automorphic forms, namely those with nonvanishing $(\mathfrak{g}, K)$-cohomology. These were classified by Vogan–Zuckermann, after prior work of Parthasarathy, Kumaresan, Enright, and Speh.

But which are these?

It turns out that there are conditions placed on the "infinity type" π_∞ of the associated automorphic representations $\pi = \pi_\infty \otimes \pi_f$. These conditions allow certain infinite-dimensional unitary representations of Lie groups to appear as π_∞ and not others.

SL_2

If $\mathbf{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$, then apart from the trivial representation we have certain well known infinite-dimensional unitary representations:

- principal series
- complementary series
- discrete series
- limits of discrete series

The condition to appear in cohomology is that π_∞ must be trivial or discrete series. Discrete series $\rightarrow$ holomorphic modular forms. Maass forms, for instance, don't show up in cohomology because they have principal series as infinity type. Similarly, weight one forms correspond to limits of discrete series.

SL_2

If $\mathbf{G}(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})$, then there are no discrete series representations.

However, there is a family of "discretely parameterized" representations.

Not surprisingly these are closely related to the modules $M_{n,n}(\mathbb{C})$ from before.

For details, see Harder's notes cohomology of arithmetic groups (available from his website).

But what about computations?

For explicit computations we need good models for the locally symmetric spaces $\Gamma\backslash X$. In particular we need to be able to apply tools from combinatorial topology (cell decompositions, . . .)

Unfortunately these are only known in a few cases, although these cases are very interesting and already very rich:

- Linear symmetric spaces, such as GL_n over number fields and division algebras, hyperbolic spaces, and a certain real form of E_6 (Voronoi, Koecher, Ash)
- Siegel upper halfspace of degree 2, i.e. $\mathbf{G} = \mathrm{Sp}_4/\mathbb{Q}$ (McConnell–MacPherson)
- Picard modular surfaces, i.e. $\mathbf{G} = \mathrm{SU}(2,1)$ over imaginary quadratic fields (Yasaki)

Linear case

We focus now on one particular example to illustrate some of aspects of the main point:

$\mathbf{G}$	$\mathrm{SL}_n/\mathbb{Q}$
G	$\mathrm{SL}_n(\mathbb{R})$
K	$\mathrm{SO}(n)$
A_G	trivial
X	cone of positive-definite real quadratic forms in n variables modulo homotheties
$\Gamma \subset \mathbf{G}(\mathbb{Q})$	congruence subgroup $\Gamma_0(N) \subset SL_n(\mathbb{Z})$ of matrices with bottom row $(0, \ldots, 0, *)$ mod N
E	trivial coefficients

Which n?

$n = 2$ classical modular forms

$n = 3$ Ash–Grayson–Green, Ash–McConnell, van Geemen–van der Kallen–Top–Verberkmoes

$n = 4$ Ash–G–McConnell

We'll focus on $n = 4$. Certain computational aspects are actually closely related to part II of the series.

Virtual cohomological dimension

Let $q = q(\mathbf{G})$ be the $\mathbb{Q}$-rank of $\mathbf{G}$, i.e. the dimension of a maximal $\mathbb{Q}$-split torus. For instance if $\mathbf{G} = \mathrm{SL}_n/\mathbb{Q}$ then $q = n - 1$.

Theorem (Borel–Serre)

For all Γ and E as above, we have $H^i(\Gamma; E) = 0$ if $i > \dim X - q$.

The number $\nu(\Gamma) = \dim X - q$ is called the *virtual cohomological dimension*.

Cuspidal range for $\mathrm{SL}_n(\mathbb{Z})$

The cuspidal cohomology doesn't appear in every cohomological degree. In fact, one can show that $H^i_{\mathrm{cusp}}(\Gamma; E) = 0$ unless the degree i lies in a small interval about $(\dim X)/2$.

n	2	3	4	5	6	7	8	9
$\dim X$	2	5	9	14	20	27	35	44
$\nu(\Gamma)$	1	3	6	10	15	21	28	36
top degree of H^*_{cusp}	1	3	5	8	11	15	19	24
bottom degree of H^*_{cusp}	1	2	4	6	9	12	16	20

Table: The virtual cohomological dimension and the cuspidal range for subgroups of $\mathrm{SL}_n(\mathbb{Z})$

Connection with arithmetic geometry

The groups $H^*(\Gamma; E)$ have an action of the *Hecke operators*, which are endomorphisms of the cohomology associated to certain finite index subgroups of Γ.

We expect eigenclasses of these operators to reveal arithmetic information in the cohomology.

Galois representations and eigenclasses

Let's get the Galois group involved.

$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$	absolute Galois group of $\mathbb{Q}$
$\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_p)$	continuous semisimple Galois representation unramified outside pN
Frob_l	Frobenius congugacy class over l

We can consider the characteristic polynomial

$$\det(1 - \rho(\mathrm{Frob}_l)T).$$

Galois representations and eigenclasses

On the cohomology side, for each prime l not dividing N we have Hecke operators $T(l,k)$, $k = 1, \ldots, n-1$. These operators generalize the classical operator T_l on modular forms.
If ξ is a Hecke eigenclass, define the *Hecke polynomial*

$$H(\xi) = \sum_k (-1)^k l^{k(k-1)/2} a(l,k) T^k \in \mathbb{C}[T].$$

where $a(l,k)$ is the eigenvalue of $T(l,k)$.

Note that if we put $T = l^{-s}$ this is the inverse of the local factor of the associated standard L-function attached to the automorphic representation (with the F.E. $s \to n-s$)

Galois representations and eigenclasses

Fix an isomorphism $\bar{\mathbb{Q}}_p \simeq \mathbb{C}$.

Conjecture

For any Hecke eigenclass ξ *of level* N*, there is a Galois representation* $\rho\colon Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Q}_p)$ *unramified outside* pN *such that for every prime* l *not dividing* pN*, we have*

$$H(\xi) = \det(1 - \rho(Frob_l)T).$$

Note that $\Gamma\backslash X$ is *not* an algebraic variety if $n > 2$, so we can't use etale cohomology to look for the Galois action. In other words, there is no direct connection to the Galois group.

This is the conjecture we're ultimately testing. We're primarily interested in *essentially nonselfdual classes*, which means that the associated automorphic representation π doesn't satisfy $\pi^\vee \simeq \pi \otimes \chi$.

Our goals

- Compute $H^5(\Gamma_0(N); \mathbb{C})$ for as big a range of levels N as possible. The degree 5 is chosen because it's in the cuspidal range, and is as close to the vcd $\nu(\Gamma)$ as possible (cf. II).
- Compute the action of the Hecke operators on this space.
- Identify Galois representations attached to the cohomology
- Try to understand whatever we can about this cohomology space.

Results

We have computed $H^5(\Gamma_0(N); \mathbb{C})$ for N prime and ≤ 211, and for composite N up to 52. The biggest computation involved matrices of size 845712×3277686 ($N = 211$).

- No nonselfdual cuspidal classes were found :(
- We found Eisenstein classes (boundary cohomology) attached to weight 2 and weight 4 modular forms.
- We found Eisenstein classes attached to SL_3 cuspidal cohomology.
- Found selfdual cuspidal classes that are apparently functorial lifts of Siegel modular forms.

For N prime we believe this is a complete description of the cohomology, apart from nonselfdual classes.

Eisenstein cohomology

$\bar{X}$ partial bordification of X due to Borel–Serre
$\Gamma\backslash\bar{X}$ Borel–Serre compactification (orbifold with corners)
$\partial(\Gamma\backslash\bar{X}) = \Gamma\backslash\bar{X} \smallsetminus \Gamma\backslash X$.
We have

$$H^*(\Gamma\backslash\bar{X};\mathbb{C}) \simeq H^*(\Gamma\backslash X;\mathbb{C}).$$

The inclusion $\partial(\Gamma\backslash\bar{X}) \hookrightarrow \Gamma\backslash\bar{X}$ induces a restriction map

$$H^*(\Gamma\backslash\bar{X};\mathbb{C}) \to H^*(\partial(\Gamma\backslash\bar{X});\mathbb{C}),$$

and *Eisenstein classes* are those restricting nontrivially to the boundary (Harder)

Weights 2 and 4

Each weight 2 eigenform f contributes to $H^5(\Gamma; \mathbb{C})$ in two different ways, with the Hecke polynomials

$$(1 - l^2 T)(1 - l^3 T)(1 - \alpha T + l T^2)$$

and

$$(1 - T)(1 - l T)(1 - l^2 \alpha T + l^5 T^2),$$

where $T_l f = \alpha f$.

A weight 4 eigenform g contributes with Hecke polynomial

$$(1 - l T)(1 - l^2 T)(1 - \beta T + l^3 T^2),$$

where $T_l g = \beta g$, if and only if the central special value of the L-function of g vanishes.

SL_3 cuspidal classes

These cohomology classes were originally computed by Ash–Grayson–Green.

An SL_3 cuspidal class with eigenvalues γ and γ' contributes in two different ways, with the Hecke polynomials

$$(1 - l^3 T)(1 - \gamma T + l\gamma' T^2 - l^3 T^3)$$

and

$$(1 - T)(1 - l\gamma T + l^3\gamma' T^2 - l^6 T^3).$$

Siegel modular forms

Let $K(p)$ be the *paramodular group* of prime level

$$K(p) = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} \\ \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} \subset \mathrm{Sp}_4(\mathbb{Q}).$$

Let $S^3(p)$ be the space of weight three paramodular forms (they are all cuspforms; there are no Eisenstein series).
This space contains the subspace $S^3_{\mathrm{G}}(p)$ of *Gritsenko lifts*, which are lifts from certain weight 3 Jacobi forms to $S^3(p)$.
Let $S^3_{\mathrm{nG}}(p)$ be the Hecke complement to $S^3_{\mathrm{G}}(p)$ in $S^3(p)$.

Siegel modular forms

The space of cuspidal paramodular forms is known pretty explicitly. First we have a dimension formula due to Ibukiyama.
Let $\kappa(a)$ be the Kronecker symbol $(\frac{a}{p})$. Define functions $f, g\colon \mathbb{Z} \to \mathbb{Q}$ by

$$f(p) = \begin{cases} 2/5 & \text{if } p \equiv 2, 3 \bmod 5, \\ 1/5 & \text{if } p = 5, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(p) = \begin{cases} 1/6 & \text{if } p \equiv 5 \bmod 12, \\ 0 & \text{otherwise.} \end{cases}$$

Ibukiyama's theorem

Theorem (Ibukiyama)

For p prime we have $\dim S^3(2) = \dim S^3(3) = 0$*. For* $p \geq 5$*, we have*

$$\begin{aligned}\dim S^3(p) = {} & (p^2-1)/2880 \\ & + (p+1)(1-\kappa(-1))/64 + 5(p-1)(1+\kappa(-1))/192 \\ & + (p+1)(1-\kappa(-3))/72 + (p-1)(1+\kappa(-3))/36 \\ & + (1-\kappa(2))/8 + f(p) + g(p) - 1.\end{aligned}$$

Using this one can easily compute the dimension of $S^3_{\mathrm{nG}}(p)$.

Hecke eigenvalues

Next, Poor and Yuen have developed a technique to compute Hecke eigenvalues for forms in $S^3_{\mathrm{nG}}(p)$.
Putting these two together, we find

- For all p, the dimension of the subspace of $H^5(\Gamma_0(p); \mathbb{C})$ not accounted for by the Eisenstein classes above matches $2 \dim S^3_{\mathrm{nG}}(p)$ according to Ibukiyama.
- In cases where we have computed the Hecke action on this subspace, we find full agreement with the data produced by Poor–Yuen.

To do

- Prove that the Eisenstein classes we see actually occur for all p.
- Prove that we do indeed have a lift from Siegel modular forms to the cohomology.
- Investigate nontrivial coefficients, torsion coefficients.

Cheers.