Automata and affine Kazhdan–Lusztig cells

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Finite state automata and regular languages

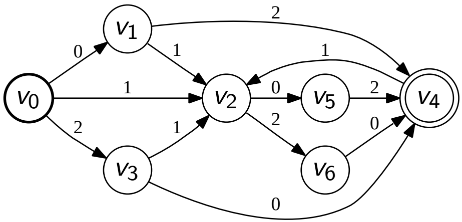
Consider an alphabet $A = \{a, b, c, \dots\}$. An *automaton over A* is a finite directed graph with some extra structure:

- Edges of the graph are labelled with symbols from A.
- There is a unique *initial* vertex.
- Some vertices are called *accepting*.

Such a graph determines a language L over A: one follows any directed path starting at the initial vertex and terminating in an accepting vertex, and builds a word by concatenating symbols along the path. A language constructed in this way is called *regular*.

Finite state automata and regular languages

Example: Let $A = \{0, 1, 2\}$.



Vertex v_0 is initial, and vertex v_4 is the only accepting vertex. Any path beginning at v_0 and ending at v_4 gives a word in the language: 02, 20, 0102, 2102, $102 \cdots 102, \ldots$. Not every word is accepted, e.g. 01.

Finite state automata and regular languages

Not every language is regular. For instance, let $A = \{0, 1\}$ and consider

$$L = \{0^k 1^k \mid k \geq 1\}.$$

Then L is not regular.

Coxeter groups

Let (W, S) be a Coxeter group. Let $\texttt{Red}(W)$ be the language of all reduced expressions of all elements of W, and let $\texttt{ShortLex}(W) \subset \texttt{Red}(W)$ be the sublanguage of "lexicographically minimal" expressions.

Theorem. [Brink–Howlett] *Both* $\texttt{Red}(W)$ *and* $\texttt{ShortLex}(W)$ *are regular.*

In fact they showed more. They proved that W is an *automatic group*.

Kazhdan–Lusztig cells

Kazhdan–Lusztig cells are subsets of W defined by an equivalence relation using descent sets and Kazhdan–Lusztig polynomials.

Given $w \in W$, let $\mathscr{L}(w) = \{s \in S \mid sw < w\}$ be the left descent set of w.

For any pair $x, y \in W$ we have a polynomial $P_{x,y}(q) \in \mathbb{Z}[q]$, the *Kazhdan–Lusztig polynomial.* We have $P_{x,y} = 0$ unless $x \leq y$ and $P_{x,x} = 1$. Otherwise the maximal possible degree of $P_{x,y}$ is $(l(y) - l(x) - 1)/2$. We write x—y if the degree of $P_{x,y}$ is maximal, and write y—x if $x < y$ and x—y.

For any x, y we can compute $P_{x,y}$ by an elementary but complicated recursion. It seems very hard to predict whether or not x—y.

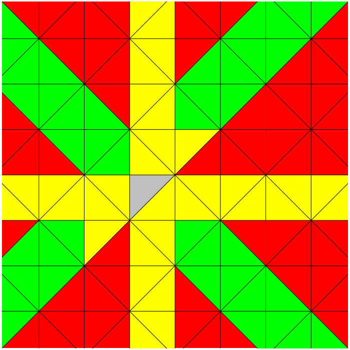
Kazhdan–Lusztig cells

Now make a directed graph $\Gamma_{\mathscr{L}}$ with vertices W and with an edge $x \to y$ if x—y and $\mathscr{L}(x) \not\subset \mathscr{L}(y)$.

The *left cells* of W are the strong connected components of the graph $\Gamma_{\mathscr{L}}$. That is, x and y are in the same left cell if there is a directed path in $\Gamma_{\mathscr{L}}$ from x to y and one from y to x.

Elements x and y are in the same *right cell* if x^{-1} and y^{-1} are in the same left cell. They're in the same *two-sided cell* if they're in the same left *or* right cell.

Example: $\tilde{C}_2$



The colors indicate the two-sided cells, and the connected sets of a given color are the left cells.

Cells as regular languages

In all known examples the cells have a simpler structure than their complicated defintion suggests. Based on this Casselman conjectured the following:

Conjecture. *For any Kazhdan–Lusztig cell $C \subset W$, the language* $\texttt{Red}(C)$ *is regular.*

Main result

Theorem. [G] *If W is an affine Weyl group then* $\mathtt{Red}(C)$ *is regular.*

Sketch of Proof

The proof uses two ingredients:

- A new family of automata recognizing $\mathtt{Red}(W)$. Vertices are certain convex unions of alcoves.
- A result of Du implying that any left cell of W can be written as a union of finitely many certain convex sets of alcoves.

The automata

Let $\{\alpha\}$ be the set of positive roots of W. For $N > 0$ we take the hyperplane arrangement $\mathscr{H}_N$ of all affine hyperplanes

$$\{H_{\alpha,k} \mid k = N, N-1, \ldots, 1-N\},$$

where

$$H_{\alpha,k} = \{x \mid \langle \alpha, x \rangle = k\}.$$

The automata

The regions in the complement of $\mathscr{H}_N$ give the vertices of the automaton. The identity alcove is the initial vertex. We connect $R \to R'$ by an edge labelled s if R and the identity alcove lie on the same side of the hyperplane determined by s and if $R \cdot s \subset R'$. If all vertices are accepting we get $\mathtt{Red}(W)$.

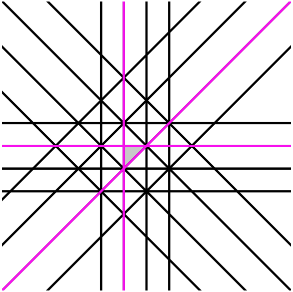


Figure: $\tilde{C}_2$, $N = 2$

Cells

Using Du's result, we can show that if N is large enough, any cell C will be a union of regions from the complement of $\mathscr{H}_N$. If we make the corresponding vertices accepting, we get $\mathtt{Red}(C)$.

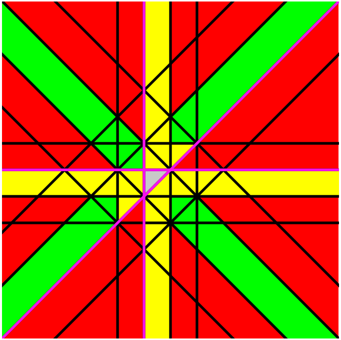
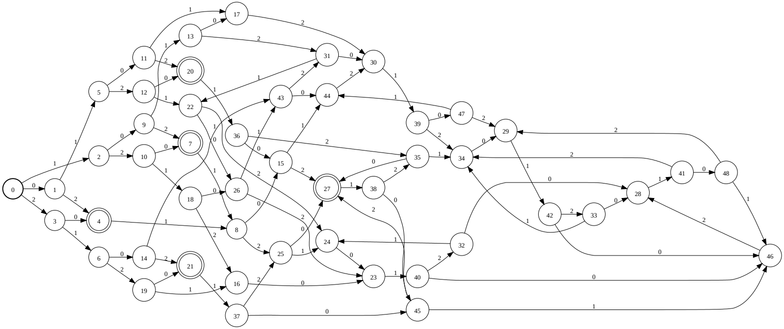


Figure: $\tilde{C}_2, N = 2$

Example

The upper left green cell. (Actually here we take a slightly smaller arrangement than $\mathscr{H}_2$, by using $N = 1$ for the short roots and $N = 2$ for the long roots.)



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General W

The same ideas won't directly work for general Coxeter groups. Nevertheless we believe Casselman's conjecture (and even have some evidence).