# Explicitly Computing Modular Forms

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Abstract. This is a textbook about algorithms for computing with modular forms. It is nontraditional in that the primary focus is not on underlying theory; instead, it answers the question *"how do you explicitly compute spaces of modular forms?"*

To my grandmother, Annette Maurer.

# Contents

# Computing in Higher Rank

by Paul E. Gunnells

## A.1. Introduction

This book has addressed the theoretical and practical problems of performing computations with modular forms. Modular forms are the simplest examples of the general theory of automorphic forms attached to a reductive algebraic group $G$ with an arithmetic subgroup $\Gamma$; they are the case $G = \mathrm{SL}_2(\mathbb{R})$ with $\Gamma$ a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For such pairs $(G, \Gamma)$ the Langlands philosophy asserts that there should be deep connections between automorphic forms and arithmetic, connections that are revealed through the action of the Hecke operators on spaces of automorphic forms. There have been many profound advances in recent years in our understanding of these phenomena, for example:

- the establishment of the modularity of elliptic curves defined over $\mathbb{Q}$ [**Wil95, TW95, Dia96, CDT99, BCDT01**],
- The proof by Harris–Taylor of the local Langlands correspondence [**HT01**], and
- Lafforgue's proof of the global Langlands correspondence for function fields [**Laf02**].

Nevertheless, we are still far from seeing that the links between automorphic forms and arithmetic hold in the broad scope in which they are generally believed. Hence one has the natural problem of studying spaces of automorphic forms computationally.

The goal of this appendix is to describe some computational techniques for automorphic forms. We focus on the case $G = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, since the automorphic forms that arise are one natural generalization of modular forms, and since this is the setting for which we have the most tools available. In fact, we don't work directly with automorphic forms, but rather with the cohomology of the arithmetic group $\Gamma$ with certain coefficient modules. This is the most natural generalization of the tools developed in previous chapters.

Here is a brief overview of the contents. Section A.2 gives background on automorphic forms and the cohomology of arithmetic groups, and explains why the two are related. In section A.3 we describe the basic topological tools used to compute the cohomology of $\Gamma$ explicitly. Section A.4 defines the Hecke operators, describes the generalization of the modular symbols from Chapter **??** to higher rank, and explains how to compute the action of the Hecke operators on the top degree cohomology group. Section A.5 discusses computation of the Hecke action on cohomology groups below the top degree. Finally, Section A.6 briefly discusses some related material, and presents some open problems.

**A.1.1.** The theory of automorphic forms is notorious for the difficulty of its prerequisites. Even if one is only interested in the cohomology of arithmetic groups—a small part of the full theory—one needs considerable background in algebraic groups, algebraic topology, and representation theory. This is somewhat reflected in our presentation, which falls far short of being self-contained. Indeed, a complete account would require a long book of its own. We have chosen to sketch the foundational material and to provide many pointers to the literature; good general references are [**BW00, Harb, LS90, Vog97**]. We hope that the energetic reader will follow the references and fill many gaps on his/her own.

The choice of topics presented here is heavily influenced (as usual) by the author's interests and expertise. There are many computational topics in the cohomology of arithmetic groups we have completely omitted, including the trace formula in its many incarnations [**GP05**], the explicit Jacquet–Langlands correspondence [**Dem04, SW05**], and moduli space techniques [**FvdG, vdG**]. We encourage the reader to investigate these extremely interesting and useful techniques.

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## A.2. Automorphic Forms and Arithmetic Groups

**A.2.1.** Let $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be the usual Hecke congruence subgroup of matrices upper-triangular mod $N$. Let $Y_0(N)$ be the modular curve $\Gamma\backslash\mathfrak{h}$, and let $X_0(N)$ be its canonical compactification obtained by adjoining cusps. For any integer $k \geq 2$, let $S_k(N)$ be the space of weight $k$ holomorphic cuspidal modular forms on $\Gamma$. According to Eichler–Shimura [**Shi94**, Chapter 8], we have the isomorphism

$$H^1(X_0(N); \mathbb{C}) \xrightarrow{\sim} S_2(N) \oplus \overline{S_2(N)}, \tag{A.2.1}$$

where the bar denotes complex conjugation, and where the isomorphism is one of Hecke modules.

More generally, for any integer $n \geq 0$, let $M_n \subset \mathbb{C}[x, y]$ be the subspace of degree $n$ homogeneous polynomials. The space $M_n$ admits a representation of $\Gamma$ by the "change of variables" map

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot p(x, y) = p(dx - by, -cx + ay). \tag{A.2.2}$$

This induces a local system $\widetilde{M}_n$ on the curve $X_0(N)$.([1]) Then the analogue of (A.2.1) for higher weight modular forms is the isomorphism

$$H^1(X_0(N); \widetilde{M}_{k-2}) \xrightarrow{\sim} S_k(N) \oplus \overline{S_k(N)}. \tag{A.2.3}$$

Note that (A.2.3) reduces to (A.2.1) when $k = 2$.

Similar considerations apply if we work with the open curve $Y_0(N)$ instead, except that Eisenstein series also contribute to the cohomology. More precisely, let $E_k(N)$ be the space of weight $k$ Eisenstein series on $\Gamma_0(N)$. Then (A.2.3) becomes

$$H^1(Y_0(N); \widetilde{M}_{k-2}) \xrightarrow{\sim} S_k(N) \oplus \overline{S_k(N)} \oplus E_k(N). \tag{A.2.4}$$

These isomorphisms lie at the heart of the modular symbol method.

**A.2.2.** The first step on the path to general automorphic forms is a reinterpretation of modular forms in terms of functions on $\mathrm{SL}_2(\mathbb{R})$. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. A weight $k$ modular form on $\Gamma$ is a holomorphic function $f\colon \mathfrak{h} \to \mathbb{C}$ satisfying the transformation property

$$f((az + b)/(cz + d)) = j(\gamma, z)^k f(z), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \quad z \in \mathfrak{h}.$$

Here $j(\gamma, z)$ is the *automorphy factor* $cz + d$. There are some additional conditions $f$ must satisfy at the cusps of $\mathfrak{h}$, but these are not so important for our discussion.

[1] The classic references for cohomology with local systems are [**Ste99a**, §31] and [**Eil47**, Ch. V]. A more recent exposition (in the language of Čech cohomology and locally constant sheaves) can be found in [**BT82**, II.13]. For an exposition tailored to our needs, see [**Harb**, §2.9].

The group $G = \mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathfrak{h}$, with the subgroup $K = \mathrm{SO}(2)$ fixing $i$. Thus $\mathfrak{h}$ can be written as the quotient $G/K$. From this, we see that $f$ can be viewed as a function $G \to \mathbb{C}$ that is $K$*-invariant on the right* and that satisfies a certain symmetry condition with respect to the $\Gamma$*-action on the left*. Of course not every $f$ with these properties is a modular form: some extra data is needed to take the role of holomorphicity and to handle the behavior at the cusps. Again, this can be ignored right now.

We can turn this interpretation around as follows. Suppose $\varphi$ is a function $G \to \mathbb{C}$ that is $\Gamma$*-invariant on the left*, that is $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$. Hence $\varphi$ can be thought of as a function $\varphi\colon \Gamma\backslash G \to \mathbb{C}$. We further suppose that $\varphi$ satisfies a certain symmetry condition with respect to the $K$*-action on the right.* In particular, any matrix $m \in K$ can be written

$$m = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}, \tag{A.2.5}$$

with $\theta$ uniquely determined modulo $2\pi$. Let $\zeta_m$ be the complex number $e^{i\theta}$. Then the $K$-symmetry we require is

$$\varphi(gm) = \zeta_m^{-k}\varphi(z), \quad m \in K,$$

where $k$ is some fixed nonnegative integer.

It turns out that such functions $\varphi$ are very closely related to modular forms: *any* $f \in S_k(\Gamma)$ uniquely determines such a function $\varphi_f\colon \Gamma\backslash G \to \mathbb{C}$. The correspondence is very simple. Given a weight $k$ modular form $f$, define

$$\varphi_f(g) := f(g\cdot i)j(g,i)^{-k}. \tag{A.2.6}$$

We claim $\varphi_f$ is left $\Gamma$-invariant and satisfies the desired $K$-symmetry on the right. Indeed, since $j$ satisfies the cocycle property $j(gh,z) = j(g, h\cdot z)j(h,z)$, we have

$$\varphi_f(\gamma g) = f((\gamma g)\cdot i)j(\gamma g,i)^{-k} = j(\gamma, g\cdot i)^k f(g\cdot i)j(\gamma, g\cdot i)^{-k}j(g,i)^{-k} = \varphi_f(g).$$

Moreover, any $m \in K$ stabilizes $i$. Hence

$$\varphi_f(gm) = f((gm)\cdot i)j(gm,i)^{-k} = f(g\cdot i)j(m,i)^{-k}j(g,m\cdot i)^{-k}.$$

From (A.2.5) we have $j(m,i)^{-k} = (\cos\theta + i\sin\theta)^{-k} = \zeta_m^{-k}$, and thus $\varphi_f(gm) = \zeta_m^{-k}\varphi_f(g)$.

Hence in (A.2.6) the weight and the automorphy factor "untwist" the $\Gamma$-action to make $\varphi_f$ left $\Gamma$-invariant. The upshot is that we can study modular forms by studying the spaces of functions that arise through the construction (A.2.6).

Of course, not every $\varphi\colon \Gamma\backslash G \to \mathbb{C}$ will arise as $\varphi_f$ for some $f \in S_K(\Gamma)$: after all, $f$ is holomorphic and satisfies rather stringent growth conditions.

Pinning down all the requirements is somewhat technical, and is (mostly) done in the sequel.

**A.2.3.** Before we define automorphic forms, we need to find the correct generalizations of our groups $\mathrm{SL}_2(\mathbb{R})$ and $\Gamma_0(N)$. The correct setup is rather technical, but this really reflects the power of the general theory, which handles so many different situations (e.g., Maass forms, Hilbert modular forms, Siegel modular forms, ...)

Let $G$ be a connected Lie group, and let $K \subset G$ be a maximal compact subgroup. We assume that $G$ is the set of real points of a connected semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$. These conditions mean the following [**PR94**, §2.1.1]:

(1) The group $\mathbf{G}$ has the structure of an affine algebraic variety given by an ideal $I$ in the ring $R = \mathbb{C}[x_{ij}, D^{-1}]$, where the variables $\{x_{ij} \mid 1 \leq i, j \leq n\}$ should be interpreted as the entries of an "indeterminate matrix," and $D$ is the polynomial $\det(x_{ij})$. Both the group multiplication $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$ and inversion $\mathbf{G} \to \mathbf{G}$ are required to be morphisms of algebraic varieties.

The ring $R$ is the coordinate ring of the algebraic group $\mathrm{GL}_n$. Hence this condition means that $\mathbf{G}$ can be essentially viewed as a subgroup of $\mathrm{GL}_n(\mathbb{C})$ defined by polynomial equations in the matrix entries of the latter.

(2) *Defined over* $\mathbb{Q}$ means that $I$ is generated by polynomials with rational coefficients.

(3) *Connected* means that $\mathbf{G}$ is connected as an algebraic variety.

(4) *Set of real points* means that $G$ is the set of real solutions to the equations determined by $I$. We write $G = \mathbf{G}(\mathbb{R})$.

(5) *Semisimple* means that the maximal connected solvable normal subgroup of $\mathbf{G}$ is trivial.

**Example A.1.** The most important example for our purposes is the *split form of* $\mathrm{SL}_n$. For this choice we have

$$G = \mathrm{SL}_n(\mathbb{R}) \text{ and } K = \mathrm{SO}(n).$$

**Example A.2.** Let $F/\mathbb{Q}$ be a number field. Then there is a $\mathbb{Q}$-group $\mathbf{G}$ such that $\mathbf{G}(\mathbb{Q}) = \mathrm{SL}_n(F)$. The group $\mathbf{G}$ is constructed as $\mathbf{R}_{F/\mathbb{Q}}(\mathrm{SL}_n)$, where $\mathbf{R}_{F/\mathbb{Q}}$ denotes the *restriction of scalars* from $F$ to $\mathbb{Q}$ [**PR94**, §2.1.2]. For example, if $F$ is totally real, the group $\mathbf{R}_{F/\mathbb{Q}}(\mathrm{SL}_2)$ appears when one studies Hilbert modular forms.

Let $(r, s)$ the signature of the field $F$, so that $F \otimes \mathbb{R} \simeq \mathbb{R}^r \times \mathbb{C}^s$. Then $G = \mathrm{SL}_n(\mathbb{R})^r \times \mathrm{SL}_n(\mathbb{C})^s$ and $K = \mathrm{SO}(n)^r \times \mathrm{SU}(n)^s$.

**Example A.3.** Another important example is the *split symplectic group* $\mathrm{Sp}_{2n}$. This is the group that arises when one studies Siegel modular forms. The group of real points $\mathrm{Sp}_{2n}(\mathbb{R})$ is the subgroup of $\mathrm{SL}_{2n}(\mathbb{R})$ preserving a fixed non-degenerate alternating bilinear form on $\mathbb{R}^{2n}$. We have $K = \mathrm{U}(n)$.

**A.2.4.** To generalize $\Gamma_0(N)$, we need the notion of an *arithmetic group*. This is a discrete subgroup $\Gamma$ of the group of rational points $\mathbf{G}(\mathbb{Q})$ that is commensurable with the set of integral points $\mathbf{G}(\mathbb{Z})$. Here commensurable simply means that $\Gamma \cap \mathbf{G}(\mathbb{Z})$ is a finite index subgroup of both $\Gamma$ and $\mathbf{G}(\mathbb{Z})$; in particular $\mathbf{G}(\mathbb{Z})$ itself is an arithmetic group.

**Example A.4.** For the split form of $\mathrm{SL}_n$ we have $\mathbf{G}(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z}) \subset \mathbf{G}(\mathbb{Q}) = \mathrm{SL}_n(\mathbb{Q})$. A trivial way to obtain other arithmetic groups is by conjugation: if $g \in \mathrm{SL}_n(\mathbb{Q})$, then $g \cdot \mathrm{SL}_n(\mathbb{Z}) \cdot g^{-1}$ is also arithmetic.

A more interesting collection of examples is given by the congruence subgroups. The *principal congruence subgroup* $\Gamma(N)$ is the group of matrices congruent to the identity modulo $N$ for some fixed integer $N \geq 1$. A *congruence subgroup* is a group containing $\Gamma(N)$ for some $N$.

In higher dimensions there are many candidates to generalize the Hecke subgroup $\Gamma_0(N)$. For example, one can take the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ that is upper-triangular mod $N$. From a computational perspective, this choice is not so good since its index in $\mathrm{SL}_n(\mathbb{Z})$ is large. A better choice, and the one that usually appears in the literature, is to define $\Gamma_0(N)$ to be the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ with bottom row congruent to $(0, \ldots, 0, *) \mod N$.

**A.2.5.** We are almost ready to define automorphic forms. Let $\mathfrak{g}$ be the Lie algebra of $G$, and let $U(\mathfrak{g})$ be its universal enveloping algebra over $\mathbb{C}$. Geometrically, $\mathfrak{g}$ is just the tangent space at the identity of the smooth manifold $G$. The algebra $U(\mathfrak{g})$ is a certain complex associative algebra canonically built from $\mathfrak{g}$. The usual definition would us a bit far afield, so we will settle for an equivalent characterization: $U(\mathfrak{g})$ can be realized as a certain subalgebra of the ring of differential operators on $C^\infty(G)$, the space of smooth functions on $G$.

In particular, $G$ acts on $C^\infty(G)$ by *left translations*: given $g \in G$ and $f \in C^\infty(G)$, we define

$$L_g(f)(x) := f(g^{-1}x).$$

Then $U(\mathfrak{g})$ can be identified with the ring of all differential operators on $C^\infty(G)$ that are invariant under left translation. For our purposes the most important part of $U(\mathfrak{g})$ is its center $Z(\mathfrak{g})$. In terms of differential operators, $Z(\mathfrak{g})$ consists of those operators that are also invariant under *right translation*:

$$R_g(f)(x) := f(xg).$$

**Definition A.5.** An *automorphic form* on $G$ with respect to $\Gamma$ is a function $\varphi\colon G \to \mathbb{C}$ satisfying

(1) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma$,

(2) the right translates $\{\varphi(gk) \mid k \in K\}$ span a finite-dimensional space $\xi$ of functions,

(3) there exists an ideal $J \subset Z(\mathfrak{g})$ of finite codimension such that $J$ annihilates $\varphi$, and

(4) $\varphi$ satisfies a certain growth condition that we do not wish to make precise. (In the literature, $\varphi$ is said to be *slowly increasing*).

For fixed $\xi$ and $J$, we denote by $\mathscr{A}(\Gamma, \xi, J, K)$ the space of all functions satisfying the above four conditions. It is a basic theorem, due to Harish-Chandra [**HC68**], that $\mathscr{A}(\Gamma, \xi, J, K)$ is finite-dimensional.

**Example A.6.** We can identify the cuspidal modular forms $S_k(N)$ in the language of Definition A.5. Given a modular form $f$, let $\varphi_f \in C^\infty(\mathrm{SL}_2(\mathbb{R}))$ be the function from (A.2.6). Then the map $f \mapsto \varphi_f$ identifies $S_k(N)$ with the subspace $\mathscr{A}_k(N)$ of functions $\varphi$ satisfying

(1) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \Gamma_0(N)$,

(2) $\varphi(gm) = \zeta_m^{-k}\varphi(g)$ for all $m \in \mathrm{SO}(2)$,

(3) $(\Delta + \lambda_k)\varphi = 0$, where $\Delta \in Z(\mathfrak{g})$ is the *Laplace–Beltrami–Casimir operator*, and
$$\lambda_k = \frac{k}{2}\left(\frac{k}{2} - 1\right),$$

(4) $\varphi$ is slowly increasing, and

(5) $\varphi$ is *cuspidal*.

The first four conditions parallel Definition A.5. Item (1) is the $\Gamma$-invariance. Item (2) implies that the right translates of $\varphi$ by $\mathrm{SO}(2)$ lie in a fixed finite-dimensional representation of $\mathrm{SO}(2)$. Item (3) is how holomorphicity appears, namely that $\varphi$ is killed by a certain differential operator. Finally, item (4) is the usual growth condition.

The only condition missing from the general definition is (5), which is an extra constraint placed on $\varphi$ to ensure that it comes from a cusp form. This condition can be expressed by the vanishing of certain integrals ("constant terms"); for details we refer to [**Bum97, Gel75**].

**Example A.7.** Another important example appears when we set $k = 0$ in (2) in Example A.6, and relax (3) by requiring only that $(\Delta - \lambda)\varphi = 0$ for *some* nonzero $\lambda \in \mathbb{R}$. Such automorphic forms cannot possibly arise from modular forms, since there are no nontrivial cusp forms of weight 0. However, there are plenty of solutions to these conditions: they correspond to *real-analytic* cuspidal modular forms of weight 0, and are known as *Maass forms*. Traditionally one writes $\lambda = (1 - s^2)/4$. The positivity of $\Delta$ implies that $s \in (-1, 1)$ or is purely imaginary.

Maass forms are highly elusive objects. Selberg proved that there are infinitely many linearly independent Maass forms of full level (i.e. on $\mathrm{SL}_2(\mathbb{Z})$), but to this date no explicit construction of a single one is known. (Selberg's argument is indirect and relies on the trace formula; for an exposition see [**Sar03**].) For higher levels some explicit examples can be constructed using theta series attached to indefinite quadratic forms [**Vig77**]. Numerically Maass forms have been well studied, see for example [**FL**].

In general the arithmetic nature of the eigenvalues $\lambda$ that correspond to Maass forms is unknown, although a famous conjecture of Selberg states that for congruence subgroups they satisfy the inequality $\lambda \geq 1/4$ (in other words, only purely imaginary $s$ appear above). The truth of this conjecture would have far-reaching consequences, from analytic number theory to graph theory [**Lub94**].

**A.2.6.** As Example A.6 indicates, there is a notion of *cuspidal automorphic form*. The exact definition is too technical to state here, but it involves an appropriate generalization of the notion of constant term familiar from modular forms.

There are also *Eisenstein series* [**Lan66, Art79**]. Again the complete definition is technical; we only mention that there are different types of Eisenstein series corresponding to certain subgroups of $G$. The Eisenstein series that are easiest to understand are those built from cusp forms on lower rank groups. Very explicit formulas for Eisenstein series on $\mathrm{GL}_3$ can be seen in [**Bum84**]. For a down-to-earth exposition of some of the Eisenstein series on $\mathrm{GL}_n$, we refer to [**Gol05**].

The decomposition of $M_k(\Gamma_0(N))$ into cusp forms and Eisenstein series also generalizes to a general group $G$, although the statement is much more complicated. The result is a theorem of Langlands [**Lan76**] known as the *spectral decomposition of* $L^2(\Gamma\backslash G)$. A thorough recent presentation of this can be found in [**MW94**].

**A.2.7.** Let $\mathscr{A} = \mathscr{A}(\Gamma, K)$ be the space of all automorphic forms, where $\xi$ and $J$ range over all possibilities. The space $\mathscr{A}$ is huge, and the arithmetic significance of much of it is unknown. This is already apparent for $G = \mathrm{SL}_2(\mathbb{R})$. The automorphic forms directly connected with arithmetic are the holomorphic modular forms, not the Maass forms.([2]) Thus the question arises: which automorphic forms in $\mathscr{A}$ are the most natural generalization of the modular forms?

[2] However, Maass forms play a very important *indirect* role in arithmetic.

One answer is provided by the isomorphisms (A.2.1), (A.2.3), (A.2.4). These show that modular forms appear naturally in the cohomology of modular curves. Hence a reasonable approach is to generalize the *left* of (A.2.1), (A.2.3), (A.2.4), and to study the resulting cohomology groups. This is the approach we will take. One drawback is that it's not obvious that our generalization has anything to do with automorphic forms, but we will see eventually that it certainly does. So we begin by looking for an appropriate generalization of the modular curve $Y_0(N)$.

Let $G$ and $K$ be as in §A.2.3, and let $X$ be the quotient $G/K$. This is a global Riemannian symmetric space [**Hel01**]. One can prove that $X$ is contractible. Any arithmetic group $\Gamma \subset G$ acts on $X$ properly discontinuously. In particular, if $\Gamma$ is torsion-free, then the quotient $\Gamma\backslash X$ is a smooth manifold.

Unlike the modular curves, $\Gamma\backslash X$ will not have a complex structure in general;([3]) nevertheless, $\Gamma\backslash X$ is a very nice space. In particular, if $\Gamma$ is torsion-free it is an *Eilenberg–Mac Lane* space for $\Gamma$, otherwise known as a $K(\Gamma, 1)$. This means that the only nontrivial homotopy group of $\Gamma\backslash X$ is its fundamental group, which is isomorphic to $\Gamma$, and that the universal cover of $\Gamma\backslash X$ is contractible. Hence $\Gamma\backslash X$ is in some sense a "topological incarnation" of $\Gamma$.([4])

This leads us to the notion of the *group cohomology* $H^*(\Gamma; \mathbb{C})$ of $\Gamma$ with trivial complex coefficients. In the early days of algebraic topology, this was defined to be the complex cohomology of an Eilenberg–Mac Lane space for $\Gamma$ [**Bro94**, Introduction, I.4]:

$$H^*(\Gamma; \mathbb{C}) = H^*(\Gamma\backslash X; \mathbb{C}). \tag{A.2.7}$$

Today there are purely algebraic approaches to $H^*(\Gamma; \mathbb{C})$ [**Bro94**, III.1], but for our purposes (A.2.7) is exactly what we need. In fact, the group cohomology $H^*(\Gamma; \mathbb{C})$ can be identified with the cohomology of the quotient $\Gamma\backslash X$ even if $\Gamma$ has torsion, since we are working with complex coefficients. The cohomology groups $H^*(\Gamma; \mathbb{C})$, where $\Gamma$ is an arithmetic group, are our proposed generalization for the weight two modular forms.

What about higher weights? For this we must replace the trivial coefficient module $\mathbb{C}$ with local systems, just as we did in (A.2.3). For our purposes it is enough to let $\mathscr{M}$ be a rational finite-dimensional representation of $G$ over the complex numbers. Any such $\mathscr{M}$ gives a representation of $\Gamma \subset G$, and thus induces a local system $\widetilde{\mathscr{M}}$ on $\Gamma\backslash X$. As before, the group cohomology $H^*(\Gamma; \mathscr{M})$ is the cohomology $H^*(\Gamma\backslash X; \widetilde{\mathscr{M}})$. In (A.2.3) we took

[3] The symmetric spaces that have a complex structure are known as *bounded domains*, or *Hermitian symmetric spaces* [**Hel01**].

[4] This apt phrase is due to Vogan [**Vog97**].

$\mathscr{M} = M_n$, the $n$th symmetric power of the standard representation. For a general group $G$ there are many kinds of representations to consider. In any case, we contend that the cohomology spaces

$$H^*(\Gamma; \mathscr{M}) = H^*(\Gamma\backslash X; \widetilde{\mathscr{M}})$$

are a good generalization of the spaces of modular forms.

**A.2.8.** It is certainly not obvious that the cohomology groups $H^*(\Gamma; \mathscr{M})$ have *anything* to do with automorphic forms, although the isomorphisms (A.2.1), (A.2.3), (A.2.4) look promising.

The connection is provided by a deep theorem of Franke [**Fra98**], which asserts that

(1) the cohomology groups $H^*(\Gamma; \mathscr{M})$ can be directly computed in terms of certain automorphic forms (the automorphic forms of "cohomological type," also known as those with "nonvanishing $(\mathfrak{g}, K)$ cohomology" [**VZ84**]); and
(2) there is a direct sum decomposition

$$H^*(\Gamma; \mathscr{M}) = H^*_{\text{cusp}}(\Gamma; \mathscr{M}) \oplus \bigoplus_{\{P\}} H^*_{\{P\}}(\Gamma; \mathscr{M}), \tag{A.2.8}$$

where the sum is taken over the set of classes of *associate proper* $\mathbb{Q}$*-parabolic subgroups of* $G$.

The precise version of statement (1) is known in the literature as the *Borel conjecture*. Statement (2) parallels Langlands's spectral decomposition of $L^2(\Gamma\backslash G)$.

**Example A.8.** For $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$, the decomposition (A.2.8) is exactly (A.2.4). The cuspforms $S_k(N) \oplus \overline{S_k(N)}$ correspond to the summand $H^1_{\text{cusp}}(\Gamma; \mathscr{M})$. There is one class of proper $\mathbb{Q}$-parabolic subgroups in $\mathrm{SL}_2(\mathbb{R})$, represented by the Borel subgroup of upper triangular matrices. Hence only one term appears in big direct sum on the right of (A.2.8), which is the Eisenstein term $E_k$.

The summand $H^*_{\text{cusp}}(\Gamma; \mathscr{M})$ of (A.2.8) is called the *cuspidal cohomology*; this is the subspace of classes represented by cuspidal automorphic forms. The remaining summands constitute the *Eisenstein cohomology* of $\Gamma$ [**Har91**]. In particular the summand indexed by $\{P\}$ is constructed using Eisenstein series attached to certain cuspidal automorphic forms on lower rank groups. Hence $H^*_{\text{cusp}}(\Gamma; \mathscr{M})$ is in some sense the most important part of the cohomology: all the rest can be built systematically from cuspidal cohomology on lower rank groups.([5]) This leads us to our basic computational problem:

**Problem A.9.** Develop tools to compute explicitly the cohomology spaces $H^*(\Gamma; \mathscr{M})$, and to identify the cuspidal subspace $H^*_{\mathrm{cusp}}(\Gamma; \mathscr{M})$.

## A.3. Combinatorial Models for Group Cohomology

**A.3.1.** In this section, we restrict attention to $G = \mathrm{SL}_n(\mathbb{R})$ and $\Gamma$ a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$. By the previous section, we can study the group cohomology $H^*(\Gamma; \mathscr{M})$ by studying the cohomology $H^*(\Gamma\backslash X; \widetilde{\mathscr{M}})$. The latter spaces can be studied using standard topological techniques, such as taking the cohomology of complexes associated to cellular decompositions of $\Gamma\backslash X$. For $\mathrm{SL}_n(\mathbb{R})$, one can construct such decompositions using a version of explicit reduction theory of real positive-definite quadratic forms due to Voronoĭ [**Vor08**]. The goal of this section is to explain how this is done. We also discuss how the cohomology can be explicitly studied for congruence subgroups of $\mathrm{SL}_3(\mathbb{Z})$.

**A.3.2.** Let $V$ be the $\mathbb{R}$-vector space of all symmetric $n \times n$ matrices, and let $C \subset V$ be the subset of positive-definite matrices. The space $C$ can be identified with the space of all real positive-definite quadratic forms in $n$ variables: in coordinates, if $x = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$ (column vector), then the matrix $A \in C$ induces the quadratic form

$$x \longmapsto x^t A x,$$

and it is well known that any positive-definite quadratic form arises in this way. The space $C$ is a cone, in that it is preserved by homotheties: if $x \in C$, then $\lambda x \in C$ for all $\lambda \in \mathbb{R}_{>0}$. It is also convex: if $x_1, x_2 \in C$, then $tx_1 + (1-t)x_2 \in C$ for $t \in [0, 1]$. Let $D$ be the quotient of $C$ by homotheties.

**Example A.10.** The case $n = 2$ is illustrative. We can take coordinates on $V \simeq \mathbb{R}^3$ by representing any matrix in $V$ as

$$\begin{pmatrix} x & y \\ y & z \end{pmatrix}, \quad x, y, z \in \mathbb{R}.$$

The subset of singular matrices $Q = \{xz - y^2 = 0\}$ is a quadric cone in $V$ dividing the complement $V \smallsetminus Q$ into three connected components. The component containing the identity matrix is the cone $C$ of positive-definite matrices. The quotient $D$ can be identified with an open 2-disk.

[5] This is a bit of an oversimplification, since it is a highly nontrivial problem to decide when cusp cohomology from lower rank groups appears in $\Gamma$. However, many results are known; as a selection we mention [**Har91, Har87, LS04**]

The group $G$ acts on $C$ on the left by

$$(g, c) \longmapsto gcg^t.$$

This action commutes with that of the homotheties, and thus descends to a $G$-action on $D$. One can show that $G$ acts transitively on $D$ and that the stabilizer of the image of the identity matrix is $K = \mathrm{SO}(n)$. Hence we may identify $D$ with our symmetric space $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$. We will do this in the sequel, using the notation $D$ when we want to emphasize the coordinates coming from the linear structure of $C \subset V$, and the notation $X$ for the quotient $G/K$.

We can make the identification $D \simeq X$ more explicit. If $g \in \mathrm{SL}_n(\mathbb{R})$, then the map

$$g \longmapsto gg^t. \tag{A.3.1}$$

takes $g$ to a symmetric positive-definite matrix. Any coset $gK$ is taken to the same matrix since $KK^t = \mathrm{Id}$. Thus (A.3.1) identifies $G/K$ with a subset $C_1$ of $C$, namely those positive-definite symmetric matrices with determinant 1. It is easy to see that $C_1$ maps diffeomorphically onto $D$.

The inverse map $C_1 \to G/K$ is more complicated. Given a determinant 1 positive-definite symmetric matrix $A$, one must find $g \in \mathrm{SL}_n(\mathbb{R})$ such that $gg^t = A$. Such a representation always exists, with $g$ determined uniquely up to right multiplication by an element of $K$. In computational linear algebra, such a $g$ can be constructed through *Cholesky decomposition* of $A$.

The group $\mathrm{SL}_n(\mathbb{Z})$ acts on $C$ via the $G$-action, and does so properly discontinuously. This is the “unimodular change of variables” action on quadratic forms [**Ser73a**, V.1.1]. Under our identification of $D$ with $X$, this is the usual action of $\mathrm{SL}_n(\mathbb{Z})$ by left translation from §A.2.7.

**A.3.3.** Now consider the group cohomology $H^*(\Gamma; \mathscr{M}) = H^*(\Gamma\backslash X; \widetilde{\mathscr{M}})$. The identification $D \simeq X$ shows that the dimension of $X$ is $n(n+1)/2 - 1$. Hence $H^i(\Gamma; \mathscr{M})$ vanishes if $i > n(n+1)/2 - 1$. Since $\dim X$ grows quadratically in $n$, there are many potentially interesting cohomology groups to study.

However, it turns out that there is some additional vanishing of the cohomology for deeper (topological) reasons. For $n = 2$, this is easy to see. The quotient $\Gamma\backslash\mathfrak{h}$ is homeomorphic to a topological surface with punctures, corresponding to the cusps of $\Gamma$. Any such surface $S$ can be retracted onto a finite graph simply by “stretching” $S$ along its punctures. Thus $H^2(\Gamma; \mathscr{M}) = 0$, even though $\dim \Gamma\backslash\mathfrak{h} = 2$.

For $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, a theorem of Borel–Serre implies that $H^i(\Gamma; \mathscr{M})$ vanishes if $i > \dim X - n + 1 = n(n-1)/2$ [**BS73**, Theorem 11.4.4]. The number $\nu = n(n-1)/2$ is called the *virtual cohomological dimension* of $\Gamma$, and is denoted $\operatorname{vcd}\Gamma$. Thus we only need to consider cohomology in degrees $i \leq \nu$.

Moreover we know from §A.2.8 that the most interesting part of the cohomology is the cuspidal cohomology. In what degrees can it live? For $n = 2$, there is only one interesting cohomology group $H^1(\Gamma; \mathscr{M})$, and it contains the cuspidal cohomology. For higher dimensions, the situation is quite different: for most $i$, the subspace $H^i_{\mathrm{cusp}}(\Gamma; \mathscr{M})$ vanishes! In fact in the late 70's Borel, Wallach, and Zuckerman observed that the cuspidal cohomology can only live in the cohomological degrees lying in an interval around $(\dim X)/2$ of size linear in $n$. An explicit description of this interval is given in [**Sch86**, Proposition 3.5]; one can also look at Table A.3.1, from which the precise statement is easy to determine.

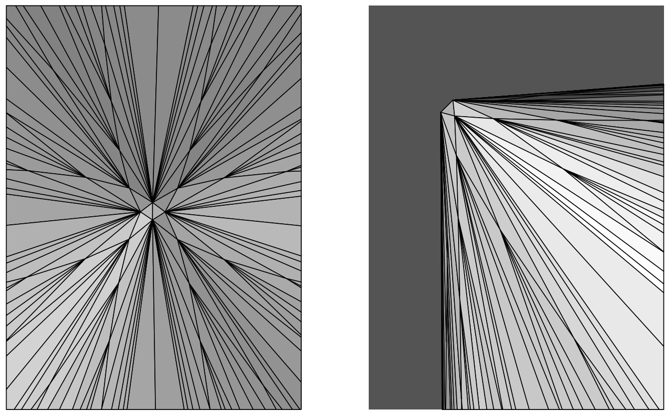
Another feature of Table A.3.1 deserves to be mentioned. There are exactly two values of $n$, namely $n = 2, 3$, such that virtual cohomological dimension equals the upper limit of the cuspidal range. This will have implications later, when we study the action of the Hecke operators on the cohomology.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|
| $\dim X$ | 2 | 5 | 9 | 14 | 20 | 27 | 35 | 44 |
| $\operatorname{vcd}\Gamma$ | 1 | 3 | 6 | 10 | 15 | 21 | 28 | 36 |
| top degree of $H^*_{\mathrm{cusp}}$ | 1 | 3 | 5 | 8 | 11 | 15 | 19 | 24 |
| bottom degree of $H^*_{\mathrm{cusp}}$ | 1 | 2 | 4 | 6 | 9 | 12 | 16 | 20 |

**Table A.3.1.** The virtual cohomological dimension and the cuspidal range for subgroups of $\mathrm{SL}_n(\mathbb{Z})$

**A.3.4.** Recall that a point in $\mathbb{Z}^n$ is said to be *primitive* if the greatest common divisor of its coordinates is 1. In particular, a primitive point is nonzero. Let $\mathscr{P} \subset \mathbb{Z}^n$ be the set of primitive points. Any $v \in \mathscr{P}$, written as a column vector, determines a rank-one symmetric matrix $q(v)$ in the closure $\bar{C}$ via $q(v) = vv^t$. The *Voronoĭ polyhedron* $\Pi$ is defined to be the closed convex hull in $\bar{C}$ of the points $q(v)$, as $v$ ranges over $\mathscr{P}$. Note that by construction, $\mathrm{SL}_n(\mathbb{Z})$ acts on $\Pi$, since $\mathrm{SL}_n(\mathbb{Z})$ preserves the set $\{q(v)\}$ and acts linearly on $V$.

**Example A.11.** Figure A.3.1 represents a crude attempt to show what $\Pi$ looks like for $n = 2$. These images were constructed by computing a large subset of the points $q(v)$ and taking the convex hull (we took all points $v \in \mathscr{P}$ such that $\operatorname{Trace} q(v) < N$ for some large integer $N$). From a distance, the polyhedron $\Pi$ looks almost indistinguishable from the cone $C$; this is somewhat conveyed by the right of Figure A.3.1. Unfortunately $\Pi$ isn't locally finite, so we really can't produce an accurate picture. To get a more accurate image, the reader should imagine that each vertex meets infinitely many edges. On the other hand, $\Pi$ isn't hopelessly complex: each maximal face is a triangle, as the pictures suggest.



(a) (b)

**Figure A.3.1.** The polyhedron $\Pi$ for $\mathrm{SL}_2(\mathbb{Z})$. In (a) we see $\Pi$ from the origin, in (b) from the side. The small triangle at the right center of (a) is the facet with vertices $\{q(e_1), q(e_2), q(e_1 + e_2)\}$, where $\{e_1, e_2\}$ is the standard basis of $\mathbb{Z}^2$. In (b) the $x$-axis runs along the top from left to right, and the $z$-axis runs down the left side. The facet from (a) is the little triangle at the top left corner of (b).

**A.3.5.** The polyhedron $\Pi$ is quite complicated: it has infinitely many faces, and is not locally finite. However, one of Voronoĭ's great insights is that $\Pi$ is actually not as complicated as it seems.

For any $A \in C$, let $\mu(A)$ be the minimum value attained by $A$ on $\mathscr{P}$ and let $M(A) \subset \mathscr{P}$ be the set on which $A$ attains $\mu(A)$. Note that $\mu(A) > 0$ and $M(A)$ is finite since $A$ is positive-definite. Then $A$ is called *perfect* if it is recoverable from the knowledge of the pair $(\mu(A), M(A))$. In other words, given $(\mu(A), M(A))$, we can write a system of linear equations

$$mZm^t = \mu(A), \quad m \in M(A), \tag{A.3.2}$$

where $Z = (z_{ij})$ is a symmetric matrix of variables. Then $A$ is perfect if and only if $A$ is the unique solution to the system (A.3.2).

**Example A.12.** The quadratic form $Q(x, y) = x^2 - xy + y^2$ is perfect. The smallest nontrivial value it attains on $\mathbb{Z}^2$ is $\mu(Q) = 1$, and it does so on the columns of

$$M(Q) = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and their negatives. Letting $\alpha x^2 + \beta xy + \gamma y^2$ be an undetermined quadratic form and applying the data $(\mu(Q), M(Q))$, we are led to the system of linear equations

$$\alpha = 1, \quad \gamma = 1, \quad \alpha + \beta + \gamma = 1.$$

From this we recover $Q(x, y)$.

**Example A.13.** The quadratic form $Q'(x, y) = x^2 + y^2$ isn't perfect. Again the smallest nontrivial value of $Q'$ on $\mathbb{Z}^2$ is $m(Q') = 1$, attained on the columns of

$$M(Q') = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and their negatives. But every member of the one-parameter family of quadratic forms

$$x^2 + \alpha xy + y^2, \quad \alpha \in (-1, 1) \tag{A.3.3}$$

has the same set of minimal vectors, and so $Q'$ can't be recovered from the knowledge of $m(Q'), M(Q')$.

**Example A.14.** Example A.12 generalizes to all $n$. Define

$$A_n(x) := \sum_{i=1}^{n} x_i^2 - \sum_{1 \le i < j \le n} x_i x_j. \tag{A.3.4}$$

Then $A_n$ is perfect for all $n$. We have $\mu(A_n) = 1$, and $M(A_n)$ consists of all points of the form

$$\pm(e_i + e_{i+1} + \cdots + e_{i+k}), \quad 1 \le i \le n, \quad i \le i + k \le n,$$

where $\{e_i\}$ is the standard basis of $\mathbb{Z}^n$. This quadratic form is closely related to the $A_n$ root lattice [**FH91**], which explains its name. It is one of two infinite families of perfect forms studied by Voronoĭ (the other is related to the $D_n$ root lattice).

We can now summarize Voronoĭ's main results:

(1) There are finitely many equivalence classes of perfect forms modulo the action of $\mathrm{SL}_n(\mathbb{Z})$. Voronoĭ even gave an explicit algorithm to determine all the perfect forms of a given dimension.

(2) The facets of $\Pi$, in other words the codimension 1 faces, are in bijection with the rank $n$ perfect quadratic forms. Under this correspondence the minimal vectors $M(A)$ determine a facet $F_A$ by taking the convex hull in $\bar{C}$ of the finite point set $\{q(m) \mid m \in M(A)\}$. Hence there are finitely many faces of $\Pi$ modulo $\mathrm{SL}_n(\mathbb{Z})$, and thus finitely many modulo any finite index subgroup $\Gamma$.

(3) Let $\mathscr{V}$ be the set of cones over the faces of $\Pi$. Then $\mathscr{V}$ is a *fan*, which means (i) if $\sigma \in \mathscr{V}$, then any face of $\sigma$ is also in $\mathscr{V}$; and (ii) if $\sigma, \sigma' \in \mathscr{V}$, then $\sigma \cap \sigma'$ is a common face of each.([6]) The fan $\mathscr{V}$ provides a reduction theory for $C$ in the following sense: any point $x \in C$ is contained in a unique $\sigma(x) \in \mathscr{V}$, and the set $\{\gamma \in \mathrm{SL}_n(\mathbb{Z}) \mid \gamma \cdot \sigma(x) = \sigma(x)\}$ is finite. Voronoĭ also gave an explicit algorithm to determine $\sigma(x)$ given $x$, the *Voronoĭ reduction algorithm*.

The number $N_{\mathrm{perf}}$ of equivalence classes of perfect forms modulo the action of $\mathrm{GL}_n(\mathbb{Z})$ grows rapidly with $n$ (Table A.3.2); the complete classification is known only for $n \leq 8$. For a list of perfect forms up to $n = 7$, see [**CS88**]. For a recent comprehensive treatment of perfect forms, with many historical remarks, see [**Mar03**].

| Dimension | $N_{\mathrm{perf}}$ | Authors |
|---|---|---|
| 2 | 1 | Voronoĭ [**Vor08**] |
| 3 | 1 | *ibid.* |
| 4 | 2 | *ibid.* |
| 5 | 3 | *ibid.* |
| 6 | 7 | Barnes [**Bar57**] |
| 7 | 33 | Jaquet-Chiffelle [**Jaq91, JC93**] |
| 8 | 10916 | Dutour–Schurmann–Vallentin [**DVS05**] |

**Table A.3.2.** The number $N_{\mathrm{perf}}$ of equivalence classes of perfect forms

**A.3.6.** Our goal now is to describe how the Voronoĭ fan $\mathscr{V}$ can be used to compute the cohomology $H^*(\Gamma; \mathscr{M})$. The idea is to use the cones in $\mathscr{V}$ to chop the quotient $D$ into pieces.

For any $\sigma \in \mathscr{V}$, let $\sigma^\circ$ be the open cone obtained by taking the complement in $\sigma$ of its proper faces. Then after taking the quotient by homotheties, the cones $\{\sigma^\circ \cap C \mid \sigma \in \mathscr{V}\}$ pass to locally closed subsets of $D$. Let $\mathscr{C}$ be the set of these images.

Any $c \in \mathscr{C}$ is a *topological cell*, i.e. is homeomorphic to an open ball, since $c$ is homeomorphic to a face of $\Pi$. Because $\mathscr{C}$ comes from the fan $\mathscr{V}$, the cells in $\mathscr{C}$ have good incidence properties: the closure in $D$ of any $c \in \mathscr{C}$ can be written as a finite disjoint union of elements of $\mathscr{C}$. Moreover, $\mathscr{C}$ is locally finite: by taking quotients of all the $\sigma^\circ$ meeting $C$, we have eliminated the open cones lying in $\bar{C}$, and it is these cones that are responsible for the failure of local finiteness of $\mathscr{V}$. We summarize these properties by saying

[6]Strictly speaking, Voronoĭ actually showed that every codimension 1 cone is contained in two top dimensional cones.

that $\mathscr{C}$ gives a *cellular decomposition* of $D$. Clearly $\mathrm{SL}_n(\mathbb{Z})$ acts on $\mathscr{C}$, since $\mathscr{C}$ is constructed using the fan $\mathscr{V}$. Thus we obtain a cellular decomposition of $\Gamma\backslash D$ for any torsion-free $\Gamma$.([7]) We call $\mathscr{C}$ the *Voronoĭ decomposition* of $D$.

Some care must be taken in using these cells to perform topological computations. The problem is that even though the individual pieces are homeomorphic to balls, and are glued together nicely, the boundaries of the closures of the pieces are not homeomorphic to spheres in general. (If they were, then the Voronoĭ decomposition would give rise to a *regular* cell complex [**CF67**], which can be used as a substitute for a simplicial or CW complex in homology computations.) Nevertheless, there is a way to remedy this.

Recall that a subspace $A$ of a topological space $B$ is a *strong deformation retract* if there is a continuous map $f\colon B\times[0,1]\to B$ such that $f(b,0)=b$, $f(b,1)\in A$, and $f(a,t)=a$ for all $a\in A$. For such pairs $A\subset B$ we have $H^*(A)=H^*(B)$. One can show that there is a strong deformation retraction from $C$ to itself equivariant under the actions of both $\mathrm{SL}_n(\mathbb{Z})$ and the homotheties, and that the image of the retraction modulo homotheties, denoted $W$, is naturally a locally finite regular cell complex of dimension $\nu$. Moreover, the cells in $W$ are in bijective, inclusion-reversing correspondence with the cells in $\mathscr{C}$. In particular, if a cell in $\mathscr{C}$ has *codimension* $d$, the corresponding cell in $W$ has *dimension* $d$. Thus, for example, the vertices of $W$ modulo $\mathrm{SL}_n(\mathbb{Z})$ are in bijection with the top dimensional cells in $\mathscr{C}$, which are in bijection with equivalence classes of perfect forms.

In the literature $W$ is called the *well-rounded retract*. The subspace $W\subset D\simeq X$ has a beautiful geometric interpretation. The quotient

$$\mathrm{SL}_n(\mathbb{Z})\backslash X=\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$$

can be interpreted as the moduli space of lattices in $\mathbb{R}^n$ modulo the equivalence relation of rotation and positive scaling. (cf. [**AG00**]; for $n=2$ one can also see [**Ser73a**, VII, Proposition 3]). Then $W$ corresponds to those lattices whose shortest nonzero vectors span $\mathbb{R}^n$. This is the origin of the name: the shortest vectors of such a lattice are "more round" than those of a generic lattice.

The space $W$ was known classically for $n=2$, and was constructed for $n\geq 3$ by Lannes and Soulé, although Soulé only published the case $n=3$ [**Sou75**]. The construction for all $n$ appears in work of Ash [**Ash80, Ash84**], who also generalized $W$ to a much larger class of groups. Explicit computations of the cell structure of $W$ have only been performed up to

[7] If $\Gamma$ has torsion, then cells in $\mathscr{C}$ can have nontrivial stabilizers in $\Gamma$, and thus $\Gamma\backslash\mathscr{C}$ should be considered as an "orbifold" cellular decomposition.

$n = 6$ [**EVGS02**]. Certainly computing $W$ explicitly for $n = 8$ seems very difficult, as Table A.3.2 indicates.

**Example A.15.** Figure A.3.2 illustrates $\mathscr{C}$ and $W$ for $\mathrm{SL}_2(\mathbb{Z})$. As in Example A.11, the polyhedron $\Pi$ is 3-dimensional, and so the Voronoĭ fan $\mathscr{V}$ has cones of dimensions $0, 1, 2, 3$. The 1-cones of $\mathscr{V}$, which correspond to the vertices of $\Pi$, pass to infinitely many points on the boundary $\partial\bar{D} = \bar{D} \smallsetminus D$. The 3-cones become triangles in $\bar{D}$ with vertices on $\partial\bar{D}$. In fact, the identifications $D \simeq \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) \simeq \mathfrak{h}$ realize $D$ as the Klein model for the hyperbolic plane, in which geodesics are represented by Euclidean line segments. Hence, the images of the 1-cones of $\mathscr{V}$ are none other than the usual cusps of $\mathfrak{h}$, and the triangles are the $\mathrm{SL}_2(\mathbb{Z})$-translates of the ideal triangle with vertices $\{0, 1, \infty\}$. These triangles form a tessellation of $\mathfrak{h}$ sometimes known as the *Farey tessellation.* The edges of the Voronoĭ are the $\mathrm{SL}_2(\mathbb{Z})$-translates of the ideal geodesic between 0 and $\infty$. After adjoining cusps and passing to the quotient $X_0(N)$, these edges become the supports of the Manin symbols from §**??** (cf. Figure **??**). This example also shows how the Voronoĭ decomposition fails to be a regular cell complex: the boundaries of the closures of the triangles in $D$ don't contain the vertices, and thus aren't homeomorphic to circles.

The virtual cohomological dimension of $\mathrm{SL}_2(\mathbb{Z})$ is 1. Hence the well-rounded retract $W$ is a graph (Figures A.3.2 and A.3.3). Note that $W$ is not a manifold. The vertices of $W$ are in bijection with the Farey triangles—each vertex lies at the center of the corresponding triangle—and the edges are in bijection with the Manin symbols. Under the map $D \to \mathfrak{h}$, the graph $W$ becomes the familiar "$\mathrm{PSL}_2$-tree" embedded in $\mathfrak{h}$, with vertices at the order 3 elliptic points (Figure A.3.3).

**A.3.7.** We now discuss the example $\mathrm{SL}_3(\mathbb{Z})$ in some detail. This example gives a good feeling for how the general situation compares to the case $n = 2$.

We begin with the Voronoĭ fan $\mathscr{V}$. The cone $C$ is 6-dimensional, and the quotient $D$ is 5-dimensional. There is one equivalence class of perfect forms modulo the action of $\mathrm{SL}_3(\mathbb{Z})$, represented by the form (A.3.4). Hence there are 12 minimal vectors; six are the columns of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}, \tag{A.3.5}$$

and the remaining six are the negatives of these. This implies that the cone $\sigma$ corresponding to this form is 6-dimensional and simplicial. The latter implies that the faces of $\sigma$ are the cones generated by $\{q(v) \mid v \in S\}$, where $S$ ranges over all subsets of (A.3.5). To get the full structure of the fan, one

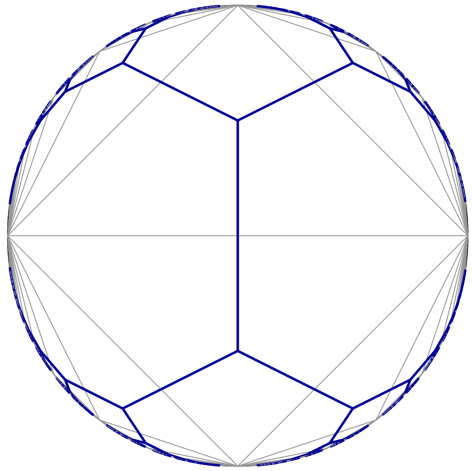


**Figure A.3.2.** The Voronoĭ decomposition and the retract in $D$

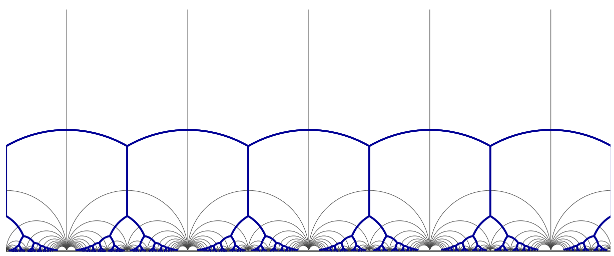


**Figure A.3.3.** The Voronoĭ decomposition and the retract in $\mathfrak{h}$

must determine the $\mathrm{SL}_3(\mathbb{Z})$ orbits of faces, as well as which faces lie in the boundary $\partial\bar{C} = \bar{C} \smallsetminus C$. After some pleasant computation, one finds:

(1) There is one equivalence class modulo $\mathrm{SL}_3(\mathbb{Z})$ for each of the 6-, 5-, 2-, and 1-dimensional cones.

(2) There are two equivalence classes of the 4-dimensional cones, represented by the sets of minimal vectors

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

(3) There are two equivalence classes of the 3-dimensional cones, represented by the sets of minimal vectors

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The second type of 3-cone lies in $\partial\bar{C}$, and thus does not determine a cell in $\mathscr{C}$.

(4) The 2- and 1-dimensional cones lie entirely in $\partial\bar{C}$, and do not determine cells in $\mathscr{C}$.

After passing from $C$ to $D$, the cones of dimension $k$ determine cells of dimension $k-1$. Therefore, modulo the action of $\mathrm{SL}_3(\mathbb{Z})$ there are five types of cells in the Voronoĭ decomposition $\mathscr{C}$, with dimensions from 5 to 2. We denote these cell types by $c_5$, $c_4$, $c_{3a}$, $c_{3b}$, and $c_2$. Here $c_{3a}$ corresponds to the first type of 4-cone in item (2) above, and $c_{3b}$ to the second. For a beautiful way to index the cells of $\mathscr{C}$ using configurations in projective spaces, see [**McC91**].

The virtual cohomological dimension of $\mathrm{SL}_3(\mathbb{Z})$ is 3, which means that the retract $W$ is a 3-dimensional cell complex. The closures of the top-dimensional cells in $W$, which are in bijection with the Voronoĭ cells of type $c_2$, are homeomorphic to solid cubes truncated along two pairs of opposite corners (Figure A.3.4). To compute this, one must see how many Voronoĭ cells of a given type contain a fixed cell of type $c_2$ (since the inclusions of cells in $W$ are the *opposite* of those in $\mathscr{C}$).

A table of the incidence relations between the cells of $\mathscr{C}$ and $W$ is given in Table A.3.3. To interpret the table, let $m = m(X, Y)$ be the integer in row $X$ and column $Y$.

- If $m$ is below the diagonal, then the boundary of a cell of type $Y$ contains $m$ cells of type $X$.
- If $m$ is above the diagonal, then a cell of type $Y$ appears in the boundary of $m$ cells of type $X$.

For instance, the entry 16 in row $c_5$ and column $c_2$ means that a Voronoĭ cell of type $c_2$ meets the boundaries of 16 cells of type $c_5$. This is same as the number of vertices in the Soulé cube (Figure A.3.4). Investigation of the table shows that the triangular (respectively, hexagonal) faces of the Soulé cube correspond to the Voronoĭ cells of type $c_{3a}$ (resp., $c_{3b}$).

Figure A.3.5 shows a Schlegel diagram for the Soulé cube. One vertex is at infinity; this is indicated by the arrows on three of the edges. This Soulé cube is dual to the Voronoĭ cell $C$ of type $c_2$ with minimal vectors given by the columns of the identity matrix. The labels on the 2-faces are additional minimal vectors that show which Voronoĭ cells contain $C$. For example, the central triangle labelled with $(1,1,1)^t$ is dual to the Voronoĭ cell of type $c_{3a}$ with minimal vectors given by those of $C$ together with $(1,1,1)^t$. Cells of type $c_4$ containing $C$ in their closure correspond to the edges of the figure; the minimal vectors for a given edge are those of $C$ together with the two vectors on the 2-faces containing the edge. Similarly, one can read off the minimial vectors of the top-dimensional Voronoĭ cells containing $C$, which correspond to the vertices of Figure A.3.5.

| | $c_5$ | $c_4$ | $c_{3a}$ | $c_{3b}$ | $c_2$ |
|---|---|---|---|---|---|
| $c_5$ | $\bullet$ | 2 | 3 | 6 | 16 |
| $c_4$ | 6 | $\bullet$ | 3 | 6 | 24 |
| $c_{3a}$ | 3 | 1 | $\bullet$ | $\bullet$ | 4 |
| $c_{3b}$ | 12 | 4 | $\bullet$ | $\bullet$ | 6 |
| $c_2$ | 12 | 8 | 4 | 3 | $\bullet$ |

**Table A.3.3.** Incidence relations in the Voronoĭ decomposition and the retract for $\mathrm{SL}_3(\mathbb{Z})$

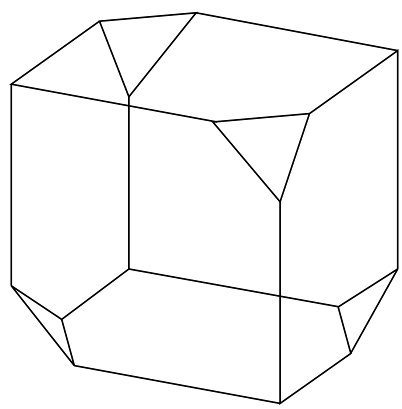


**Figure A.3.4.** The Soulé cube

**A.3.8.** Now let $p$ be a prime, and let $\Gamma = \Gamma_0(p) \subset \mathrm{SL}_3(\mathbb{Z})$ be the Hecke subgroup of matrices with bottom row congruent to $(0, 0, *) \mod p$ (Example A.4). The virtual cohomological dimension of $\Gamma$ is 3, and the cusp cohomology with constant coefficients can appear in degrees 2 and 3. One can show that the cusp cohomology in degree 2 is dual to that in degree 3, so for computational purposes it suffices to focus on degree 3.

In terms of $W$, these will be cochains supported on the 3-cells. Unfortunately we cannot work directly with the quotient $\Gamma\backslash W$ since $\Gamma$ has torsion: there will be cells taken to themselves by the $\Gamma$-action, and thus the cells of $W$ need to be subdivided to induce the structure of a cell complex on $\Gamma\backslash W$. Thus when $\Gamma$ has torsion, the "set of 3-cells modulo $\Gamma$" unfortunately makes no sense.

To circumvent this problem, one can mimic the idea of Manin symbols. The quotient $\Gamma\backslash \mathrm{SL}_3(\mathbb{Z})$ is in bijection with the finite projective plane $\mathbb{P}^2(\mathbb{F}_p)$,

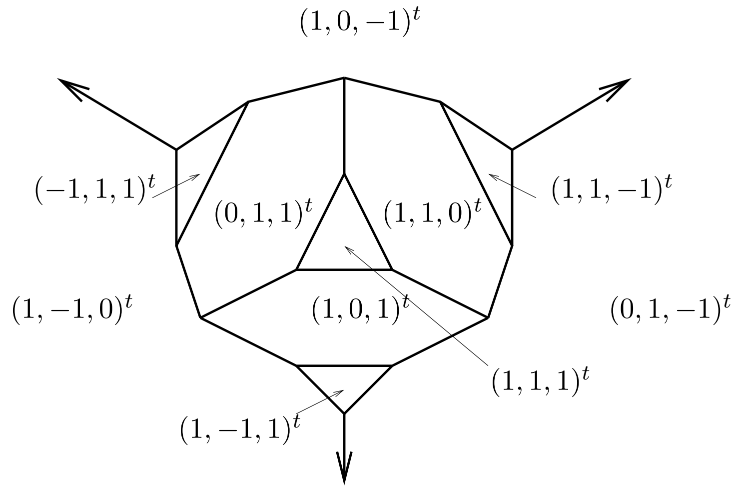



**Figure A.3.5.** A Schlegel diagram of a Soulé cube, showing the minimal vectors that correspond to the 2-faces

where $\mathbb{F}_p$ is the field with $p$ elements (cf. Proposition **??**). The group $\mathrm{SL}_3(\mathbb{Z})$ acts transitively on the set of all 3-cells of $W$; if we fix one such cell $w$, its stabilizer $\mathrm{Stab}(w) = \{\gamma \in \mathrm{SL}_3(\mathbb{Z}) \mid \gamma w = w\}$ is a finite subgroup of $\mathrm{SL}_3(\mathbb{Z})$. Hence the set of 3-cells modulo $\Gamma$ should be interpreted as the set of orbits in $\mathbb{P}^2(\mathbb{F}_p)$ of the finite group $\mathrm{Stab}(w)$. This suggests describing $H^3(\Gamma;\mathbb{C})$ in terms of the space $\mathscr{S}$ of complex-valued functions $f\colon \mathbb{P}^2(\mathbb{F}_p) \to \mathbb{C}$. To carry this out, there are two problems:

(1) How do we explicitly describe $H^3(\Gamma;\mathbb{C})$ in terms of $\mathscr{S}$?
(2) How can we isolate the cuspidal subspace $H^3_{\mathrm{cusp}}(\Gamma;\mathbb{C}) \subset H^3(\Gamma;\mathbb{C})$ in terms of our description?

Fully describing the solutions to these problems is rather complicated. We content ourselves with presenting the following theorem, which collects together several statements in [**AGG84**]. This result should be compared to Theorems **??** and **??**.

**Theorem A.16.** [**AGG84**, Theorem 3.19, Summary 3.23] *We have*

$$\dim H^3(\Gamma_0(p);\mathbb{C}) = \dim H^3_{cusp}(\Gamma_0(p);\mathbb{C}) + 2S_p,$$

*where $S_p$ is the dimension of the space of weight 2 holomorphic cusp forms on $\Gamma_0(p) \subset \mathrm{SL}_2(\mathbb{Z})$. Moreover, the cuspidal cohomology $H^3_{cusp}(\Gamma_0(p);\mathbb{C})$ is isomorphic to the vector space of functions $f\colon \mathbb{P}^2(\mathbb{F}_p) \to \mathbb{C}$ satisfying*

(1) $f(x,y,z) = f(z,x,y) = f(-x,y,z) = -f(y,x,z)$,
(2) $f(x,y,z) + f(-y,x-y,z) + f(y-x,-x,z) = 0$,
(3) $f(x,y,0) = 0$, *and*

(4) $\sum_{z=1}^{p-1} f(x, y, z) = 0$.

Unlike subgroups of $\mathrm{SL}_2(\mathbb{Z})$, cuspidal cohomology is apparently much rarer for $\Gamma_0(p) \subset \mathrm{SL}_3(\mathbb{Z})$. The computations of [**AGG84, vGvdKTV97**] show that the only prime levels $p \leq 337$ with nonvanishing cusp cohomology are 53, 61, 79, 89, and 223. In all these examples, the cuspidal subspace is 2-dimensional.

For more details of how to implement such computations, we refer to [**AGG84, vGvdKTV97**]. For further details about the additional complications arising for higher rank groups, in particular subgroups of $\mathrm{SL}_4(\mathbb{Z})$, see [**AGM02**, §3].

## A.4. Hecke Operators and Modular Symbols

**A.4.1.** There is one ingredient missing so far in our discussion of the cohomology of arithmetic groups, namely the *Hecke operators*. These are an essential tool in the study of modular forms. Indeed, the forms with the most arithmetic significance are the Hecke eigenforms, and the connection with arithmetic is revealed by the Hecke eigenvalues.

In higher rank the situation is similar. There is an algebra of Hecke operators acting on the cohomology spaces $H^*(\Gamma; \mathscr{M})$. The eigenvalues of these operators are conjecturally related to certain representations of the Galois group. Just as in the case $G = \mathrm{SL}_2(\mathbb{R})$, we need tools to compute the Hecke action.

In this section we discuss this problem. We begin with a general description of the Hecke operators and how they act on cohomology. Then we focus on one particular cohomology group, namely the top degree $H^\nu(\Gamma; \mathbb{C})$, where $\nu = \mathrm{vcd}(\Gamma)$ and $\Gamma$ has finite index in $\mathrm{SL}_n(\mathbb{Z})$. This is the setting that generalizes the modular symbol method from Chapter **??**. We conclude by giving examples of Hecke eigenclasses in the cuspidal cohomology of $\Gamma_0(p) \subset \mathrm{SL}_3(\mathbb{Z})$.

**A.4.2.** Let $g \in \mathrm{SL}_n(\mathbb{Q})$. The group $\Gamma' = \Gamma \cap g^{-1}\Gamma g$ has finite index in both $\Gamma$ and $g^{-1}\Gamma g$. The element $g$ determines a diagram $C(g)$

$$\begin{tikzcd} & \Gamma'\backslash X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \Gamma\backslash X & & \Gamma\backslash X \end{tikzcd}$$

called a *Hecke correspondence*. The map $s$ is induced by the inclusion $\Gamma' \subset \Gamma$, while $t$ is induced by the inclusion $\Gamma' \subset g^{-1}\Gamma g$ followed by the diffeomorphism $g^{-1}\Gamma g\backslash X \to \Gamma\backslash X$ given by left multiplication by $g$. Specifically,

$$s(\Gamma' x) = \Gamma x, \quad t(\Gamma' x) = \Gamma g x, \quad x \in X.$$

The maps $s$ and $t$ are finite-to-one, since the indices $[\Gamma' : \Gamma]$ and $[\Gamma' : g^{-1}\Gamma g]$ are finite. This implies that we obtain maps on cohomology

$$s^*\colon H^*(\Gamma\backslash X) \to H^*(\Gamma'\backslash X), \quad t_*\colon H^*(\Gamma'\backslash X) \to H^*(\Gamma\backslash X).$$

Here the map $s^*$ is the usual induced map on cohomology, while the "wrong-way" map $t_*$ is given by summing a class over the finite fibers of $t$.([8]) These maps can be composed to give a map

$$T_g := t_* s^*\colon H^*(\Gamma\backslash X; \widetilde{\mathscr{M}}) \longrightarrow H^*(\Gamma\backslash X; \widetilde{\mathscr{M}}),$$

This is called the *Hecke operator* associated to $g$. There is an obvious notion of isomorphism of Hecke correspondences. One can show that up to isomorphism, the correspondence $C(g)$ and thus the Hecke operator $T_g$ depend only on the double coset $\Gamma g \Gamma$. One can compose Hecke correspondences, and thus we obtain an algebra of operators acting on the cohomology, just as in the classical case.

**Example A.17.** Let $n = 2$, and let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. If we take $g = \mathrm{diag}(1, p)$, where $p$ is a prime, then the action of $T_g$ on $H^1(\Gamma; M_{k-2})$ is the same as the action of the classical Hecke operator $T_p$ on the weight $k$ holomorphic modular forms. If we take $\Gamma = \Gamma_0(N)$, we obtain an operator $T(p)$ for all $p$ prime to $N$, and the algebra of Hecke operators coincides with the (semisimple) Hecke algebra generated by the $T_p$, $(p, N) = 1$. For $p|N$, one can also describe the $U_p$ operators in this language.

**Example A.18.** Now let $n > 2$ and let $\Gamma = \mathrm{SL}_n(\mathbb{Z})$. The picture is very similar, except that now there are several Hecke operators attached to any prime $p$. In fact there are $n-1$ operators $T(p, k)$, $k = 1, \dots, n-1$. The operator $T(p, k)$ is associated to the correspondence $C(g)$, where $g = \mathrm{diag}(1, \dots, 1, p, \dots, p)$, and where $p$ occurs $k$ times. If we consider the congruence subgroups $\Gamma_0(N)$, we have operators $T(p, k)$ for $(p, N) = 1$, and analogues of the $U_p$ operators for $p|N$.

Just as in the classical case, any double coset $\Gamma g \Gamma$ can be written as a disjoint union of left cosets

$$\Gamma g \Gamma = \coprod_{h \in \Omega} \Gamma h$$

[8] Under the identification $H^*(\Gamma\backslash X; \widetilde{\mathscr{M}}) \simeq H^*(\Gamma; \mathscr{M})$, the map $t_*$ becomes the transfer map in group cohomology [**Bro94**, III.9].

for a certain finite set of $n \times n$ integral matrices $\Omega$. For the operator $T(p,k)$, the set $\Omega$ can be taken to be all upper-triangular matrices of the form [**Kri90**, Proposition 7.2]

$$\begin{pmatrix} p^{e_1} & & a_{ij} \\ & \ddots & \\ & & p^{e_n} \end{pmatrix},$$

where

- $e_i \in \{0,1\}$, and exactly $k$ of the $e_i$ are equal to 1, and
- $a_{ij} = 0$ unless $e_i = 0$ and $e_j = 1$, in which case $a_{ij}$ satisfies $0 \leq a_{ij} < p$.

**Remark A.19.** The number of coset representatives for the operator $T(p,k)$ is the same as the number of points in the finite Grassmannian $G(k,n)(\mathbb{F}_p)$. A similar phenomenon is true for the Hecke operators for any group $G$, although there are some subtleties [**Gro98**].

**A.4.3.** Recall that in §A.3.6 we constructed the Voronoĭ decomposition $\mathscr{C}$ and the well-rounded retract $W$, and that we can use them to compute the cohomology $H^*(\Gamma; \mathscr{M})$. Unfortunately, we can't directly use them to compute the action of the Hecke operators on cohomology, since the Hecke operators do not act cellularly on $\mathscr{C}$ or $W$. The problem is that the Hecke image of a cell in $\mathscr{C}$ (or $W$) is usually not a union of cells in $\mathscr{C}$ (or $W$). This is already apparent for $n = 2$. The edges of $\mathscr{C}$ are the $\mathrm{SL}_2(\mathbb{Z})$-translates of the ideal geodesic $\tau$ from 0 to $\infty$ (Example A.15). Applying a Hecke operator takes such an edge to a union of ideal geodesics, each with vertices at a pair of cusps. In general such geodesics aren't an $\mathrm{SL}_2(\mathbb{Z})$-translate of $\tau$.

For $n = 2$, one solution is to work with all possible ideal geodesics with vertices at the cusps, in other words the space of modular symbols $\mathbb{M}_2$ from §**??**. Manin's trick (Proposition **??**) shows how to write any modular symbol as a linear combination of unimodular symbols, by which we mean modular symbols supported on the edges of $\mathscr{C}$. These are the ideas we now generalize to all $n$.

**Definition A.20.** Let $S_0$ be the $\mathbb{Q}$-vector space spanned by the symbols $\mathbf{v} = [v_1, \ldots, v_n]$, where $v_i \in \mathbb{Q}^n \smallsetminus \{0\}$, modulo the following relations:

(1) If $\tau$ is a permutation on $n$ letters, then

$$[v_1, \ldots, v_n] = \mathrm{sign}(\tau)[\tau(v_1), \ldots, \tau(v_n)],$$

where $\mathrm{sign}(\tau)$ is the sign of $\tau$.

(2) If $q \in \mathbb{Q}^\times$, then

$$[qv_1, v_2 \ldots, v_n] = [v_1, \ldots, v_n].$$

(3) If the points $v_1, \dots, v_n$ are linearly dependent, then $\mathbf{v} = 0$.

Let $B \subset S_0$ be the subspace generated by linear combinations of the form

$$\sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n], \tag{A.4.1}$$

where $v_0, \dots, v_n \in \mathbb{Q}^n \smallsetminus \{0\}$, and where $\hat{v}_i$ means to omit $v_i$.

We call $S_0$ the space of *modular symbols*. We caution the reader that there are some differences in what we call modular symbols and those found in §**??** and Definition **??**; we compare them in §A.4.4. The group $\mathrm{SL}_n(\mathbb{Q})$ acts on $S_0$ by left multiplication: $g \cdot \mathbf{v} = [gv_1, \dots, gv_n]$. This action preserves the subspace $B$, and thus induces an action on the quotient $M = S_0/B$. For $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ a finite index subgroup, let $M_\Gamma$ be the space of $\Gamma$-coinvariants in $M$. In other words, $M_\Gamma$ is the quotient of $M$ by the subspace generated by $\{m - \gamma \cdot m \mid \gamma \in \Gamma\}$.

The relationship between modular symbols and the cohomology of $\Gamma$ is given by the following theorem, first proved for $\mathrm{SL}_n$ by Ash–Rudolph [**AR79**] and by Ash for general $G$ [**Ash86**]:

**Theorem A.21.** [**Ash86, AR79**] *Let $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ be a finite index subgroup. There is an isomorphism*

$$M_\Gamma \xrightarrow{\sim} H^\nu(\Gamma; \mathbb{Q}), \tag{A.4.2}$$

*where $\Gamma$ acts trivially on $\mathbb{Q}$, and where $\nu = \mathrm{vcd}(\Gamma)$.*

We remark that Theorem A.21 remains true if $\mathbb{Q}$ is replaced with nontrivial coefficients as in §A.2.7. Moreover, if $\Gamma$ is assumed to be torsion-free then we can replace $\mathbb{Q}$ with $\mathbb{Z}$.

The great virtue of $M_\Gamma$ is that it admits an action of the Hecke operators. Given a Hecke operator $T_g$, write the double coset $\Gamma g \Gamma$ as a disjoint union of left cosets

$$\Gamma g \Gamma = \coprod_{h \in \Omega} \Gamma h \tag{A.4.3}$$

as in Example A.18. Any class in $M_\Gamma$ can be lifted to a representative $\eta = \sum q(\mathbf{v})\mathbf{v} \in S_0$, where $q(\mathbf{v}) \in \mathbb{Q}$ and almost all $q(\mathbf{v})$ vanish. Then we define

$$T_g(\mathbf{v}) = \sum_{h \in \Omega} h \cdot \mathbf{v}, \tag{A.4.4}$$

and extend to $\eta$ by linearity. The right of (A.4.4) depends on the choices of $\eta$ and $\Omega$, but after taking quotients and coinvariants we obtain a well-defined action on cohomology via (A.4.2).

**A.4.4.** The space $S_0$ is closely related to the space $\mathbb{M}_2$ from §**??** and §**??**. Indeed, $\mathbb{M}_2$ was defined to be the quotient $(F/R)/(F/R)_{\mathrm{tor}}$, where $F$ is the free abelian group generated by ordered pairs

$$\{\alpha, \beta\}, \quad \alpha, \beta \in \mathbb{P}^1(\mathbb{Q}), \tag{A.4.5}$$

and $R$ is the subgroup generated by elements of the form

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\}, \quad \alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q}). \tag{A.4.6}$$

The only new feature in Definition A.20 is item (3). For $n = 2$ this corresponds to the condition $\{\alpha, \alpha\} = 0$, which follows from (A.4.6). We have

$$S_0/B \simeq \mathbb{M}_2 \otimes \mathbb{Q}.$$

Hence there are two differences between $S_0$ and $\mathbb{M}_2$: our notion of modular symbols uses rational coefficients instead of integral coefficients, and is the space of symbols *before* dividing out by the subspace of relations $B$; we further caution the reader that this is somewhat at odds with the literature.

We also remark that the general arbitrary weight definition of modular symbols for a subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ given in §**??** also includes taking $\Gamma$-coinvariants, as well as extra data for a coefficient system. We haven't included the latter data since our emphasis is trivial coefficients, although it would be easy to do so in the spirit of §**??**

Elements of $\mathbb{M}_2$ also have a geometric interpretation: the symbol $\{\alpha, \beta\}$ corresponds to the ideal geodesic in $\mathfrak{h}$ with endpoints at the cusps $\alpha$ and $\beta$. We have a similar picture for the symbols $\mathbf{v} = [v_1, \ldots, v_n]$. We can assume that each $v_i$ is primitive, which means that each $v_i$ determines a vertex of the Voronoĭ polyhedron $\Pi$. The rational cone generated by these vertices determines a subset $\Delta(\mathbf{v}) \subset D$, where $D$ is the linear model of the symmetric space $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$ from §A.3.2. This subset $\Delta(\mathbf{v})$ is then an "ideal simplex" in $X$. There is also a connection between $\Delta(\mathbf{v})$ and torus orbits in $X$; we refer to [**Ash86**] for a related discussion.

**A.4.5.** Now we need a generalization of the Manin trick (§**??**). This is known in the literature as the *modular symbol algorithm*.

We can define a kind of norm function on $S_0$ as follows. Let $\mathbf{v} = [v_1, \ldots, v_n]$ be a modular symbol. For each $v_i$, choose $\lambda_i \in \mathbb{Q}^\times$ such that $\lambda_i v_i$ is primitive. Then we define

$$\|\mathbf{v}\| := |\det(\lambda_1 v_1, \ldots, \lambda_n v_n)| \in \mathbb{Z}.$$

Note that $\|\mathbf{v}\|$ is well defined, since the $\lambda_i$ are unique up to sign, and permuting the $v_i$ only changes the determinant by a sign. We extend $\|\ \|$ to all of $S_0$ by taking the maximum of $\|\ \|$ over the support of any $\eta \in S_0$: if

$\eta = \sum q(\mathbf{v})\mathbf{v}$, where $q(\mathbf{v}) \in \mathbb{Q}$ and almost all $q(\mathbf{v})$ vanish, then we put

$$\|\eta\| = \operatorname*{Max}_{q(\mathbf{v})\neq 0} \|\mathbf{v}\|.$$

We say a modular symbol $\eta$ is *unimodular* if $\|\eta\| = 1$. It is clear that the images of the unimodular symbols generate a finite-dimensional subspace of $M_\Gamma$. The next theorem shows that this subspace is actually *all* of $M_\Gamma$.

**Theorem A.22.** [**AR79, Bar94**] *The space $M_\Gamma$ is spanned by the images of the unimodular symbols. More precisely, given any symbol $\mathbf{v} \in S_0$ with $\|\mathbf{v}\| > 1$,*

(1) *in $S_0/B$ we may write*

$$\mathbf{v} = \sum q(\mathbf{w})\mathbf{w}, \quad q(\mathbf{w}) \in \mathbb{Z}, \tag{A.4.7}$$

*where if $q(\mathbf{w}) \neq 0$ then $\|\mathbf{w}\| = 1$, and*

(2) *the number of terms on the right of* (A.4.7) *is bounded by a polynomial in* $\log\|\mathbf{v}\|$ *that depends only on the dimension $n$.*

**Proof.** (Sketch) Given a modular symbol $\mathbf{v} = [v_1, \dots, v_n]$, we may assume that the points $v_i$ are primitive. We will show that if $\|\mathbf{v}\| > 1$, we can find a point $u$ such that when we apply the relation (A.4.1) using the points $u, v_1, \dots, v_n$, all terms other than $\mathbf{v}$ have norm less than $\|\mathbf{v}\|$. We call such a point a *reducing point* for $\mathbf{v}$.

Let $P \subset \mathbb{R}^n$ be the open parallelotope

$$P := \Big\{ \sum \lambda_i v_i \;\Big|\; |\lambda_i| < \|\mathbf{v}\|^{-1/n} \Big\}.$$

Then $P$ is an $n$-dimensional centrally symmetric convex body with volume $2^n$. By Minkowski's theorem from the geometry of numbers (cf. [**FT93**, IV.2.6]), $P \cap \mathbb{Z}^n$ contains a nonzero point $u$. Using (A.4.1) we find

$$\mathbf{v} = \sum_{i=1}^{n} (-1)^{i-1} \mathbf{v}_i(u), \tag{A.4.8}$$

where $\mathbf{v}_i(u)$ is the symbol

$$\mathbf{v}_i(u) = [v_1, \dots, v_{i-1}, u, v_{i+1}, \dots, v_n].$$

Moreover, it is easy to see that the new symbols satisfy

$$0 \leq \|\mathbf{v}_i(u)\| < \|\mathbf{v}\|^{(n-1)/n}, \quad i = 1, \dots, n. \tag{A.4.9}$$

This completes the proof of the first statement.

To prove the second statement, we must estimate how many times relations of the form (A.4.8) need to be applied to obtain (A.4.7). A nonunimodular symbol produces at most $n$ new modular symbols after (A.4.8) is performed; we potentially have to apply (A.4.8) again to each of the symbols that result, which in turn could produce as many as $n$ new symbols for each. Hence we can visualize the process of constructing (A.4.7) as building a rooted tree, where the root is $\mathbf{v}$, the leaves are the symbols $\mathbf{w}$, and where each node has at most $n$ children. It is not hard to see that the bound (A.4.9) implies that the depth of this tree (i.e. the longest length of a path from the root to a leaf) is $O(\log\log\|\mathbf{v}\|)$. From this the second statement follows easily. $\square$

Statement (1) of Theorem A.22 is due to Ash–Rudolph [**AR79**]. Instead of $P$, they used the larger parallelotope $P'$ defined by

$$P' := \Big\{ \sum \lambda_i v_i \;\Big|\; |\lambda_i| < 1 \Big\},$$

which has volume $2^n\|\mathbf{v}\|$. The observation that $P'$ can be replaced by $P$, as well as the proof of (2), are both due to Barvinok [**Bar94**].

**A.4.6.** The relationship between Theorem A.22 and Manin's trick should be clear. For $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, the Manin symbols correspond exactly to the unimodular symbols mod $\Gamma$. So Theorem A.22 implies that every modular symbol (in the language of §**??**) is a linear combination of Manin symbols. This is exactly the conclusion of Proposition **??**.

In higher rank the relationship between Manin symbols and unimodular symbols is more subtle. In fact there are two possible notions of "Manin symbol," which agree for $\mathrm{SL}_2(\mathbb{Z})$ but not in general. One possibility is the obvious one: a Manin symbol is a unimodular symbol.

The other possibility is to define a Manin symbol to be a modular symbol corresponding to a top-dimensional cell of the retract $W$. But for $n \geq 5$, such modular symbols need not be unimodular. In particular, for $n = 5$ there are two equivalence classes of top-dimensional cells. One class corresponds to the unimodular symbols, the other to a set of modular symbols of norm 2. However, Theorems A.21 and A.22 show that $H^\nu(\Gamma;\mathbb{Q})$ is spanned by unimodular symbols. Thus as far as this cohomology group is concerned, the second class of symbols is in some sense unnecessary.

**A.4.7.** We return to the setting of §A.3.8 and give examples of Hecke eigenclasses in the cusp cohomology of $\Gamma = \Gamma_0(p) \subset \mathrm{SL}_3(\mathbb{Z})$. We closely follow [**AGG84, vGvdKTV97**]. Note that since the top of the cuspidal range for $\mathrm{SL}_3$ is the same as the virtual cohomological dimension $\nu$, we can use modular symbols to compute the Hecke action on cuspidal classes.

Given a prime $l$ coprime to $p$, there are two Hecke operators of interest $T(l,1)$ and $T(l,2)$. We can compute the action of these operators on

$H^3_{\mathrm{cusp}}(\Gamma;\mathbb{C})$ as follows. Recall that $H^3_{\mathrm{cusp}}(\Gamma;\mathbb{C})$ can be identified with a certain space of functions $f\colon \mathbb{P}^2(\mathbb{F}_p)\to\mathbb{C}$ (Theorem A.16). Given $x\in\mathbb{P}^2(\mathbb{F}_p)$, let $Q_x\in\mathrm{SL}_3(\mathbb{Z})$ be a matrix such that $Q_x\mapsto x$ under the identification $\mathbb{P}^2(\mathbb{F}_p)\xrightarrow{\sim}\Gamma\backslash\mathrm{SL}_3(\mathbb{Z})$. Then $Q_x$ determines a unimodular symbol $[Q_x]$ by taking the $v_i$ to be the columns of $Q_x$. Given any Hecke operator $T_g$, we can find coset representatives $h_i$ such that $\Gamma g\Gamma=\coprod\Gamma h_i$ (explicit representatives for $\Gamma=\Gamma_0(p)$ and $T_g=T(l,k)$ are given in [**AGG84, vGvdKTV97**]). The modular symbols $[h_iQ_x]$ are no longer unimodular in general, but we can apply Theorem A.22 to write

$$[h_iQ_x]=\sum_j[R_{ij}],\quad R_{ij}\in\mathrm{SL}_3(\mathbb{Z}).$$

Then for $f\colon\mathbb{P}^2(\mathbb{F}_p)\to\mathbb{C}$ as in Theorem A.16, we have

$$(T_gf)(x)=\sum_{i,j}f(\overline{R_{ij}}),$$

where $\overline{R_{ij}}$ is the class of $R_{ij}$ in $\mathbb{P}^2(\mathbb{F}_p)$.

Now let $\xi\in H^3_{\mathrm{cusp}}(\Gamma;\mathbb{C})$ be a simultaneous eigenclass for all the Hecke operators $T(l,1)$, $T(l,2)$, as $l$ ranges over all primes coprime with $p$. General considerations from the theory of automorphic forms imply that the eigenvalues $a(l,1)$, $a(l,2)$ are complex conjugates of one other. Hence it suffices to compute $a(l,1)$. We give two examples of cuspidal eigenclasses for two different prime levels.

**Example A.23.** Let $p=53$. Then $H^3_{\mathrm{cusp}}(\Gamma_0(53);\mathbb{C})$ is 2-dimensional. Let $\eta=(1+\sqrt{-11})/2$. One eigenclass is given by the data

| $l$ | 2 | 3 | 5 | 7 | 11 | 13 |
|---|---|---|---|---|---|---|
| $a(l,1)$ | $-1-2\eta$ | $-2+2\eta$ | 1 | $-3$ | 1 | $-2-12\eta$ |

and the other is obtained by complex conjugation.

**Example A.24.** Let $p=61$. Then $H^3_{\mathrm{cusp}}(\Gamma_0(61);\mathbb{C})$ is 2-dimensional. Let $\omega=(1+\sqrt{-3})/2$. One eigenclass is given by the data

| $l$ | 2 | 3 | 5 | 7 | 11 | 13 |
|---|---|---|---|---|---|---|
| $a(l,1)$ | $1-2\omega$ | $-5+4\omega$ | $-2+4\omega$ | $-6\omega$ | $-2+2\omega$ | $-2-4\omega$ |

and the other is obtained by complex conjugation.

## A.5. Other Cohomology Groups

**A.5.1.** In §A.4 we saw how to compute the Hecke action on the top cohomology group $H^\nu(\Gamma;\mathbb{C})$. Unfortunately for $n\geq 4$, this cohomology group does not contain any cuspidal cohomology. The first case is $\Gamma\subset\mathrm{SL}_4(\mathbb{Z})$; we have $\mathrm{vcd}(\Gamma)=6$, and the cusp cohomology lives in degrees 4 and 5. One can show that the cusp cohomology in degree 4 is dual to that in degree 5, so for computational purposes it suffices to be able to compute the Hecke action on $H^5(\Gamma;\mathbb{C})$. But modular symbols don't help us here.

In this section we describe a technique to compute the Hecke action on $H^{\nu-1}(\Gamma;\mathbb{C})$, following [**Gun00a**]. The technique is an extension of the modular symbol algorithm to these cohomology groups. In principle the ideas in this section can be modified to compute the Hecke action on other cohomology groups $H^{\nu-k}(\Gamma;\mathbb{C})$, $k>1$, although this has not been investigated.([9]) For $n=4$, we have applied the algorithm in joint work with Ash and McConnell to investigate computationally the cohomology $H^5(\Gamma;\mathbb{C})$, where $\Gamma_0(N)\subset \mathrm{SL}_4(\mathbb{Z})$ [**AGM02**].

**A.5.2.** To begin, we need an analogue of Theorem A.21 for lower degree cohomology groups. In other words, we need a generalization of the modular symbols for other cohomology groups. This is achieved by the *sharbly complex* $S_*$:

**Definition A.25.** [**Ash94**] Let $\{S_*,\partial\}$ be the chain complex given by the following data:

(1) For $k\geq 0$, $S_k$ is the $\mathbb{Q}$-vector space generated by the symbols $\mathbf{u}=[v_1,\dots,v_{n+k}]$, where $v_i\in\mathbb{Q}^n\smallsetminus\{0\}$, modulo the relations:
   (a) If $\tau$ is a permutation on $(n+k)$ letters, then
   $$[v_1,\dots,v_{n+k}]=\mathrm{sign}(\tau)[\tau(v_1),\dots,\tau(v_{n+k})],$$
   where $\mathrm{sign}(\tau)$ is the sign of $\tau$.
   (b) If $q\in\mathbb{Q}^\times$, then
   $$[qv_1,v_2\dots,v_{n+k}]=[v_1,\dots,v_{n+k}].$$
   (c) If the rank of the matrix $(v_1,\dots,v_{n+k})$ is less than $n$, then $\mathbf{u}=0$.

(2) For $k>0$, the boundary map $\partial\colon S_k\to S_{k-1}$ is
$$[v_1,\dots,v_{n+k}]\longmapsto\sum_{i=1}^{n+k}(-1)^i[v_1,\dots,\hat{v}_i,\dots,v_{n+k}].$$
We define $\partial$ to be identically zero on $S_0$.

The elements $\mathbf{u}=[v_1,\dots,v_{n+k}]$ are called *k-sharblies*.([10]) The 0-sharblies are exactly the modular symbols from Definition A.20, and the subspace $B\subset S_0$ is the image of the boundary map $\partial\colon S_1\to S_0$.

[9]The first interesting case is $n=5$, for which the cuspidal cohomology lives in $H^{\nu-2}$.

[10]The terminology for $S_*$ is due to Lee Rudolph, in honor of Lee and Szczarba. They introduced a very similar complex in [**LS76**] for $\mathrm{SL}_3(\mathbb{Z})$.

There is an obvious left action of $\Gamma$ on $S_*$ commuting with $\partial$. For any $k \geq 0$, let $S_{k,\Gamma}$ be the space of $\Gamma$-coinvariants. Since the boundary map $\partial$ commutes with the $\Gamma$-action, we obtain a complex $(S_{*,\Gamma}, \partial_\Gamma)$. The following theorem shows that this complex computes the cohomology of $\Gamma$:

**Theorem A.26.** [**Ash94**] *There is a natural isomorphism*

$$H^{\nu-k}(\Gamma;\mathbb{C}) \xrightarrow{\sim} H_k(S_{*,\Gamma} \otimes \mathbb{C}).$$

**A.5.3.** We can extend our norm function $\| \;\|$ from modular symbols to all of $S_k$ as follows. Let $\mathbf{u} = [v_1, \ldots, v_{n+k}]$ be a $k$-sharbly, and let $Z(\mathbf{u})$ be the set of all submodular symbols determined by $\mathbf{u}$. In other words, $Z(\mathbf{u})$ consists of the modular symbols of the form $[v_{i_1}, \ldots, v_{i_n}]$, where $\{i_1, \ldots, i_n\}$ ranges over all $n$-fold subsets of $\{1, \ldots, n+k\}$. Define $\|\mathbf{u}\|$ by

$$\|\mathbf{u}\| = \operatorname*{Max}_{\mathbf{v}\in Z(\mathbf{u})} \|\mathbf{v}\|.$$

Note that $\|\mathbf{u}\|$ is well defined modulo the relations in Definition A.25. As for modular symbols, we extend the norm to sharbly chains $\xi = \sum q(\mathbf{u})\mathbf{u}$ taking the maximum norm over the support. Formally, we let $\operatorname{supp}(\xi) = \{\mathbf{u} \mid q(\mathbf{u}) \neq 0\}$ and $Z(\xi) = \cup_{\mathbf{u}\in\operatorname{supp}(\xi)} Z(\mathbf{u})$, and then define $\|\xi\|$ by

$$\|\xi\| = \operatorname*{Max}_{\mathbf{v}\in Z(\xi)} \|\mathbf{v}\|.$$

We say that $\xi$ is *reduced* if $\|\xi\| = 1$. Hence $\xi$ is reduced if and only if all its submodular symbols are unimodular or have determinant 0. Clearly there are only finitely many reduced $k$-sharblies modulo $\Gamma$ for any $k$.

In general the cohomology groups $H^*(\Gamma;\mathbb{C})$ are *not* spanned by reduced sharblies. However, it is known (cf. [**McC91**]) that for $\Gamma \subset \mathrm{SL}_4(\mathbb{Z})$, the group $H^5(\Gamma;\mathbb{C})$ is spanned by reduced 1-sharbly cycles. The best one can say in general is that for each pair $n, k$, there is an integer $N = N(n,k)$ such that for $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, $H^{\nu-k}(\Gamma;\mathbb{C})$ is spanned by $k$-sharblies of norm $\leq N$. This set of sharblies is also finite modulo $\Gamma$, although it is not known how large $N$ must be for any given pair $n, k$.

**A.5.4.** Recall that the cells of the well-rounded retract $W$ are indexed by sets of primitive vectors in $\mathbb{Z}^n$. Since any primitive vector determines a point in $\mathbb{Q}^n \smallsetminus \{0\}$, and since sets of such points index sharblies, it is clear that there is a close relationship between $S_*$ and the chain complex associated to $W$, although of course $S_*$ is much bigger. In any case, both complexes compute $H^*(\Gamma;\mathbb{C})$.

The main benefit of using the sharbly complex to compute cohomology is that it admits a Hecke action. Suppose $\xi = \sum q(\mathbf{u})\mathbf{u}$ be a sharbly cycle mod $\Gamma$, and consider a Hecke operator $T_g$. Then we have

$$T_g(\xi) = \sum_{h\in\Omega,\mathbf{u}} n(\mathbf{u})h\cdot\mathbf{u}, \tag{A.5.1}$$

where $\Omega$ is a set of coset representatives as in (A.4.3). Since $\Omega \not\subset \mathrm{SL}_n(\mathbb{Z})$ in general, the Hecke-image of a reduced sharbly isn't usually reduced.

**A.5.5.** We are now ready to describe our algorithm for the computation of the Hecke operators on $H^{\nu-1}(\Gamma;\mathbb{C})$. It suffices to describe an algorithm that takes as input a 1-sharbly cycle $\xi$ and produces as output a cycle $\xi'$ with

(a) the classes of $\xi$ and $\xi'$ in $H^{\nu-1}(\Gamma;\mathbb{C})$ the same, and
(b) $\|\xi'\| < \|\xi\|$ if $\|\xi\| > 1$.

We will present an algorithm satisfying (a) below. In [**Gun00a**], we conjectured (and presented evidence) that the algorithm satisfies (b) for $n \leq 4$. Further evidence is provided by the computations in [**AGM02**], which relied on the algorithm to compute the Hecke action on $H^5(\Gamma;\mathbb{C})$, where $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$.

The idea behind the algorithm is simple: given a 1-sharbly cycle $\xi$ that isn't reduced, (i) simultaneously apply the modular symbol algorithm (Theorem A.22) to each of its submodular symbols, then (ii) package the resulting data into a new 1-sharbly cycle. Our experience in presenting this algorithm is that most people find the geometry involved in (ii) daunting. Hence we will give details only for $n = 2$, and will provide a sketch for $n > 2$. Full details are contained in [**Gun00a**]. Note that $n = 2$ is topologically and arithmetically uninteresting, since we're computing the Hecke action on $H^0(\Gamma;\mathbb{C})$; nevertheless, the geometry faithfully represents the situation for all $n$.

**A.5.6.** Fix $n = 2$, let $\xi \in S_1$ be a 1-sharbly cycle mod $\Gamma$ for some $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, and suppose $\xi$ is not reduced. Assume $\Gamma$ is torsion-free to simplify the presentation.

Suppose first that all submodular symbols $\mathbf{v} \in Z(\xi)$ are nonunimodular. Select reducing points for each $\mathbf{v} \in Z(\xi)$, and make these choices $\Gamma$-equivariantly. This means the following. Suppose $\mathbf{u}, \mathbf{u}' \in \operatorname{supp}\xi$ and $\mathbf{v} \in \operatorname{supp}(\partial\mathbf{u})$ and $\mathbf{v}' \in \operatorname{supp}(\partial\mathbf{u}')$ are modular symbols such that $\mathbf{v} = \gamma\cdot\mathbf{v}'$ for some $\gamma \in \Gamma$. Then we select reducing points $w$ for $\mathbf{v}$ and $w'$ for $\mathbf{v}'$ such that $w = \gamma\cdot w'$. (Note that since $\Gamma$ is torsion-free, no modular symbol can be identified to itself by an element of $\Gamma$, hence $\mathbf{v} \neq \mathbf{v}'$.) This is possible since if $\mathbf{v}$ is a modular symbol and $w$ is a reducing point for $\mathbf{v}$, then $\gamma\cdot w$ is a reducing point for $\gamma\cdot\mathbf{v}$) for any $\gamma \in \Gamma$. Because there are only finitely many

$\Gamma$-orbits in $Z(\xi)$, we can choose reducing points $\Gamma$-equivariantly by selecting them for some set of orbit representatives.

It is important to note that $\Gamma$-equivariance is the only global criterion we use when selecting reducing. In particular, there is a priori no relationship among the 3 reducing points chosen for any $\mathbf{u} \in \operatorname{supp} \xi$.

**A.5.7.** Now we want to use the reducing points and the 1-sharblies in $\xi$ to build $\xi'$. Choose $\mathbf{u} = [v_1, v_2, v_3] \in \operatorname{supp} \xi$, and denote the reducing point for $[v_i, v_j]$ by $w_k$, where $\{i, j, k\} = \{1, 2, 3\}$. We use the $v_i$ and the $w_i$ to build a 2-sharbly chain $\eta(\mathbf{u})$ as follows.

Let $P$ be an octahedron in $\mathbb{R}^3$. Label the vertices of $P$ with the $v_i$ and $w_i$ such that the vertex labeled $v_i$ is opposite the vertex labeled $w_i$ (Figure A.5.1). Subdivide $P$ into four tetrahedra by connecting two opposite vertices, say $v_1$ and $w_1$, with an edge (Figure A.5.2). For each tetrahedron $T$, take the labels of four vertices and arrange them into a quadruple. If we orient $P$, then we can use the induced orientation on $T$ to order the four primitive points. In this way, each $T$ determines a 2-sharbly, and $\eta(\mathbf{u})$ is defined to be the sum. For example, if we use the decomposition in Figure A.5.2, we have

(A.5.2)

$$\eta(\mathbf{u}) = [v_1, v_3, v_2, w_1] + [v_1, w_2, v_3, w_1] + [v_1, w_3, w_2, w_1] + [v_1, v_2, w_3, w_1].$$

Repeat this construction for all $\mathbf{u} \in \operatorname{supp} \xi$, and let $\eta = \sum q(\mathbf{u})\eta(\mathbf{u})$. Finally, let $\xi' = \xi + \partial\eta$.

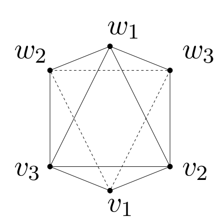



**Figure A.5.1.**

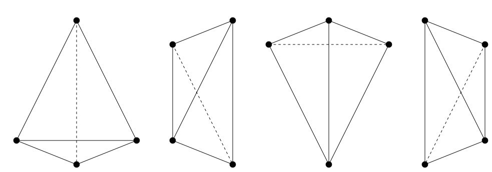


**Figure A.5.2.**

**A.5.8.** By construction, $\xi'$ is a cycle mod $\Gamma$ in the same class as $\xi$. We claim in addition that no submodular symbol from $\xi$ appears in $\xi'$. To see this, consider $\partial\eta(\mathbf{u})$. From (A.5.2), we have

$$\begin{aligned}\text{(A.5.3)}\quad \partial\eta(\mathbf{u}) = &-[v_1, v_2, v_3] + [v_1, v_2, w_3] + [v_1, w_2, v_3] + [w_1, v_2, v_3]\\ &- [v_1, w_2, w_3] - [w_1, v_2, w_3] - [w_1, w_2, v_3] + [w_1, w_2, w_3].\end{aligned}$$

Note that this is the boundary in $S_*$, not in $S_{*,\Gamma}$. Furthermore, $\partial\eta(\mathbf{u})$ is independent of which pair of opposite vertices of $P$ we connected to build $\eta(\mathbf{u})$.

From (A.5.3), we see that in $\xi + \partial\eta$ the 1-sharbly $-[v_1, v_2, v_3]$ is canceled by $\mathbf{u} \in \operatorname{supp}\xi$. We also claim that 1-sharblies in (A.5.3) of the form $[v_i, v_j, w_k]$ vanish in $\partial_\Gamma\eta$.

To see this, let $\mathbf{u}$,$\mathbf{u}' \in \operatorname{supp}\xi$, and suppose $\mathbf{v} = [v_1, v_2] \in \operatorname{supp}\partial\mathbf{u}$ equals $\gamma \cdot \mathbf{v}'$ for some $\mathbf{v}' = [v_1', v_2'] \in \operatorname{supp}\partial\mathbf{u}'$. Since the reducing points were chosen $\Gamma$-equivariantly, we have $w = \gamma \cdot w'$. This means that the 1-sharbly $[v_1, v_2, w] \in \partial\eta(\mathbf{u})$ will be canceled mod $\Gamma$ by $[v_1', v_2', w'] \in \partial\eta(\mathbf{u}')$. Hence, in passing from $\xi$ to $\xi'$, the effect in $(S_*)_\Gamma$ is to replace $\mathbf{u}$ with *four* 1-sharblies in $\operatorname{supp}\xi'$:

$$\text{(A.5.4)}\quad [v_1, v_2, v_3] \longmapsto -[v_1, w_2, w_3] - [w_1, v_2, w_3] - [w_1, w_2, v_3] + [w_1, w_2, w_3].$$

Note that in (A.5.4), there are no 1-sharblies of the form $[v_i, v_j, w_k]$.

**Remark A.27.** For implementation purposes, it is not necessary to explicitly construct $\eta$. Rather, one may work directly with (A.5.4).

**A.5.9.** Why do we expect $\xi'$ to satisfy $\|\xi'\| < \|\xi\|$? First of all, in the right hand side of (A.5.4) there are no submodular symbols of the form $[v_i, v_j]$. In fact, any submodular symbol involving a point $v_i$ also includes a reducing point for $[v_i, v_j]$.

On the other hand, consider the submodular symbols in (A.5.4) of the form $[w_i, w_j]$. Since there is no relationship among the $w_i$, one has no reason to believe that these modular symbols are closer to unimodularity than those in $\mathbf{u}$. Indeed, for certain choices of reducing points it can happen that $\|[w_i, w_j]\| \geq \|\mathbf{u}\|$.

The upshot is that some care must be taken in choosing reducing points. In [**Gun00a**, Conjectures 3.5 and 3.6] we describe two methods for finding reducing points for modular symbols, one using Voronoĭ reduction and one using *LLL*-reduction. Our experience is that if one selects reducing points using either of these conjectures, then $\|[w_i, w_j]\| < \|\mathbf{u}\|$ for each of the new modular symbols $[w_i, w_j]$. In fact, in practice these symbols are trivial or satisfy $\|[w_i, w_j]\| = 1$.

**A.5.10.** In the previous discussion we assumed that no submodular symbols of any $\mathbf{u} \in \operatorname{supp} \xi$ were unimodular. Now we say what to do if some are. There are three cases to consider.

First, all submodular symbols of $\mathbf{u}$ may be unimodular. In this case there are no reducing points, and (A.5.4) becomes

$$[v_1, v_2, v_3] \longmapsto [v_1, v_2, v_3]. \tag{A.5.5}$$

Second, one submodular symbol of $\mathbf{u}$ may be nonunimodular, say the symbol $[v_1, v_2]$. In this case to build $\eta$ we use a tetrahedron $P'$ and put $\eta(\mathbf{u}) = [v_1, v_2, v_3, w_3]$ (Figure A.5.3). Since $[v_1, v_2, w_3]$ vanishes in the boundary of $\eta$ mod $\Gamma$, (A.5.4) becomes

$$[v_1, v_2, v_3] \mapsto -[v_1, v_3, w_3] + [v_2, v_3, w_3]. \tag{A.5.6}$$

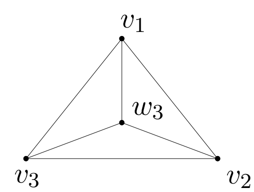



**Figure A.5.3.**

Finally, two submodular symbols of $\mathbf{u}$ may be nonunimodular, say $[v_1, v_2]$ and $[v_1, v_3]$. In this case we use the cone on a square $P''$ (Figure A.5.4). To construct $\eta(\mathbf{u})$ we must choose a decomposition of $P''$ into tetrahedra. Since $P''$ has a non-simplicial face this choice affects $\xi'$ (in contrast to the previous cases). If we subdivide $P''$ by connecting the vertex labelled $v_2$ with the vertex labelled $w_2$, we obtain

$$[v_1, v_2, v_3] \longmapsto [v_2, w_2, w_3] + [v_2, v_3, w_2] + [v_1, v_3, w_2]. \tag{A.5.7}$$

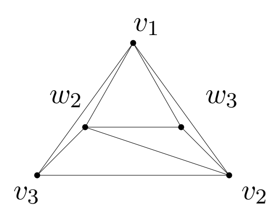



**Figure A.5.4.**

**A.5.11.** Now consider general $n$. The basic technique is the same, but the combinatorics become more complicated. Suppose $\mathbf{u} = [v_1, \ldots, v_{n+1}]$ satisfies $q(\mathbf{u}) \neq 0$ in a 1-sharbly cycle $\xi$, and for $i = 1, \ldots, n+1$ let $\mathbf{v}_i$ be the submodular symbol $[v_1, \ldots, \widehat{v_i}, \ldots, v_{n+1}]$. Assume that all $\mathbf{v}_i$ are nonunimodular, and for each $i$ let $w_i$ be a reducing point for $\mathbf{v}_i$.

For any subset $I \subset \{1, \ldots, n+1\}$, let $\mathbf{u}_I$ be the 1-sharbly $[u_1, \ldots, u_{n+1}]$, where $u_i = w_i$ if $i \in I$, and $u_i = v_i$ otherwise. The polytope $P$ used to build $\eta(\mathbf{u})$ is the *cross polytope*, which is the higher-dimensional analogue of the octahedron [**Gun00a**, §4.4]. We suppress the details and give the final answer: (A.5.4) becomes

$$\mathbf{u} \longmapsto -\sum_I (-1)^{\#I} \mathbf{u}_I, \tag{A.5.8}$$

where the sum is taken over all subsets $I \subset \{1, \ldots, n+1\}$ of of cardinality *at least* 2.

More generally, if some $\mathbf{v}_i$ happen to be unimodular, then the polytope used to build $\eta$ is an iterated cone on a lower-dimensional cross polytope. This is already visible for $n = 2$:

- The 2-dimensional cross polytope is a square, and the polytope $P''$ is a cone on a square.
- The 1-dimensional cross polytope is an interval, and the polytope $P'$ is a double cone on an interval.

Altogether there are $n + 1$ relations generalizing (A.5.5)–(A.5.7).

**A.5.12.** Now we describe how these computations are carried out in practice, focusing on $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$ and $H^5(\Gamma; \mathbb{C})$. Besides discussing technical details, we also have to slightly modify some aspects of the construction in §A.5.6, since $\Gamma$ isn't torsion-free.

Let $W$ be the well-rounded retract. We can represent a cohomology class $\beta \in H^5(\Gamma; \mathbb{C})$ as $\beta = \sum q(\sigma)\sigma$, where $\sigma$ denotes a codimension 1 cell in $W$. In this case there are three types of codimension 1 cells in $W$. Under the bijection $W \leftrightarrow \mathscr{C}$, these cells correspond to the Voronoĭ cells indexed by the columns of the matrices

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}. \tag{A.5.9}$$

Thus each $\sigma$ in $W$ modulo $\Gamma$ corresponds to an $\mathrm{SL}_4(\mathbb{Z})$-translate of one of (A.5.9). These translates determine basis 1-sharblies $\mathbf{u}$ (by taking the points

$u_i$ to be the columns), and hence we can represent $\beta$ by a 1-sharbly chain $\xi = \sum q(\mathbf{u})\mathbf{u} \in S_1$ that is a cycle in the complex of coinvariants $(S_{*,\Gamma}, \partial_\Gamma)$.

To make later computations more efficient, we precompute more data attached to $\xi$. Given a 1-sharbly $\mathbf{u} = [u_1, \dots, u_{n+1}]$, a *lift* $M(\mathbf{u})$ of $\mathbf{u}$ is defined to be an integral matrix with primitive columns $M_i$ such that $\mathbf{u} = [M_1, \dots, M_{n+1}]$. Then we encode $\xi$, once and for all, by a finite collection $\Phi$ of 4-tuples

$$(\mathbf{u}, n(\mathbf{u}), \{\mathbf{v}\}, \{M(\mathbf{v})\}),$$

where

(1) $\mathbf{u}$ ranges over the support of $\xi$,

(2) $n(\mathbf{u}) \in \mathbb{C}$ is the coefficient of $\mathbf{u}$ in $\xi$,

(3) $\{\mathbf{v}\}$ is the set of submodular symbols appearing in the bounday of $\mathbf{u}$, and

(4) $\{M(\mathbf{v})\}$ is a set of lifts for $\{\mathbf{v}\}$.

Moreover, the lifts in (4) are chosen to satisfy the following $\Gamma$-equivariance condition. Suppose that for $\mathbf{u}, \mathbf{u}' \in \operatorname{supp} \xi$ we have $\mathbf{v} \in \operatorname{supp}(\partial \mathbf{u})$ and $\mathbf{v}' \in \operatorname{supp}(\partial \mathbf{u}')$ satisfying $\mathbf{v} = \gamma \cdot \mathbf{v}'$ for some $\gamma \in \Gamma$. Then we require $M(\mathbf{v}) = \gamma M(\mathbf{v}')$. This is possible since $\xi$ is a cycle modulo $\Gamma$, although there is one complication since $\Gamma$ has torsion: it can happen that some submodular symbol $\mathbf{v}$ of a 1-sharbly $\mathbf{u}$ is identified to *itself* by an element of $\Gamma$. This means that in constructing $\{M(\mathbf{v})\}$ for $\mathbf{u}$, we must somehow choose more than one lift for $\mathbf{v}$. To deal with this, let $M(\mathbf{v})$ be any lift of $\mathbf{v}$, and let $\Gamma(\mathbf{v}) \subset \Gamma$ be the stabilizer of $\mathbf{v}$. Then in $\xi$, we replace $q(\mathbf{u})\mathbf{u}$ by

$$\frac{1}{\#\Gamma(\mathbf{v})} \sum_{\gamma \in \Gamma(\mathbf{v})} q(\mathbf{u})\mathbf{u}_\gamma,$$

where $\mathbf{u}_\gamma$ has the same data as $\mathbf{u}$, except that we give $\mathbf{v}$ the lift $\gamma M(\mathbf{v})$.([11])

Next we compute and store the 1-sharbly transformation laws generalizing (A.5.5)–(A.5.7). As a part of this we fix triangulations of certain cross polytopes as in (A.5.7).

We are now ready to begin the actual reduction algorithm. We take a Hecke operator $T(l, k)$ and build the coset representatives $\Omega$ as in (A.5.1). For each $h \in \Omega$ and each 1-sharbly $\mathbf{u}$ in the support of $\xi$, we obtain a nonreduced 1-sharbly $\mathbf{u}_h := h \cdot \mathbf{u}$. Here $h$ acts on all the data attached to $\mathbf{u}$ in the list $\Phi$. In particular, we replace each lift $M(\mathbf{v})$ with $h \cdot M(\mathbf{v})$, where the dot means matrix multiplication.

[11] In fact, we can be slightly more clever than this and only introduce denominators that are powers of 2.

Now we check the submodular symbols of $\mathbf{u}_h$ and choose reducing points for the nonunimodular symbols. This is where the lifts come in handy. Recall that reduction points must be chosen $\Gamma$-equivariantly over the entire cycle. Instead of explicitly keeping track of the identifications between modular symbols, we do the following trick:

(1) Construct the *Hermite normal form* $M_{\mathrm{her}}(\mathbf{v})$ of the lift $M(\mathbf{v})$ (see [**Coh93**, §2.4] and Exercise **??.??**). Record the transformation matrix $U \in \mathrm{GL}_4(\mathbb{Z})$ such that $UM(\mathbf{v}) = M_{\mathrm{her}}(\mathbf{v})$.
(2) Choose a reducing point $u$ for $M_{\mathrm{her}}(\mathbf{v})$.
(3) Then the reducing point for $M(\mathbf{v})$ is $U^{-1}u$.

This guarantees $\Gamma$-equivariance: if $\mathbf{v}$, $\mathbf{v}'$ are submodular symbols of $\xi$ with $\gamma \cdot \mathbf{v} = \mathbf{v}'$ and with reducing points $u, u'$, we have $\gamma u = u'$. The reason is that the Hermite normal form $M_{\mathrm{her}}(\mathbf{v})$ is a *uniquely determined* representative of the $\mathrm{GL}_4\,\mathbb{Z})$-orbit of $M(\mathbf{v})$ [**Coh93**]. Hence if $\gamma M(\mathbf{v}) = M(\mathbf{v}')$ then $M_{\mathrm{her}}(\mathbf{v}) = M_{\mathrm{her}}(\mathbf{v}')$.

After computing all reducing points, we apply the appropriate transformation law. The result will be a chain of 1-sharblies, each of which has (conjecturally) smaller norm than the original 1-sharbly $\mathbf{u}$. We output these 1-sharblies if they're reduced; otherwise they're fed into the reduction algorithm again. Eventually we obtain a reduced 1-sharbly cycle $\xi'$ homologous to the original cycle $\xi$.

The final step of the algorithm is to rewrite $\xi'$ as a cocycle on $W$. This is easy to do since the relevant cells of $W$ are in bijection with the reduced 1-sharblies. There are some nuisances in keeping orientations straight, but the computation is not difficult. We refer to [**AGM02**] for details.

**A.5.13.** We now give some examples, taken from [**AGM02**], of Hecke eigenclasses in $H^5(\Gamma_0(N);\mathbb{C})$ for various levels $N$. Instead of giving a table of eigenvalues, we give the *Hecke polynomials*. If $\beta$ is an eigenclass with $T(l,k)(\beta) = a(l,k)\beta$, then we define

$$H(\beta, l) = \sum_k (-1)^k l^{k(k-1)/2} a(l,k) X^k \in \mathbb{C}[X].$$

For almost all $l$, after putting $X = l^{-s}$ where $s$ is a complex variable, the function $H(\beta, s)$ is the inverse of the local factor at $l$ of the automorphic representation attached to $\beta$.

**Example A.28.** Suppose $N = 11$. Then the cohomology $H^5(\Gamma_0(11);\mathbb{C})$ is 2-dimensional. There are two Hecke eigenclasses $u_1, u_2$, each with rational Hecke eigenvalues.

| | | |
|---|---|---|
| $u_1$ | $T_2$ | $(1-4X)(1-8X)(1+2X+2X^2)$ |
| | $T_3$ | $(1-9X)(1-27X)(1+X+3X^2)$ |
| | $T_5$ | $(1-25X)(1-125X)(1-X+5X^2)$ |
| | $T_7$ | $(1-49X)(1-343X)(1+2X+7X^2)$ |
| $u_2$ | $T_2$ | $(1-X)(1-2X)(1+8X+32X^2)$ |
| | $T_3$ | $(1-X)(1-3X)(1+9X+243X^2)$ |
| | $T_5$ | $(1-X)(1-5X)(1-25X+3125X^2)$ |
| | $T_7$ | $(1-X)(1-7X)(1+98X+16807X^2)$ |

**Example A.29.** Suppose $N = 19$. Then the cohomology $H^5(\Gamma_0(19);\mathbb{C})$ is 3-dimensional. There are three Hecke eigenclasses $u_1, u_2, u_3$, each with rational Hecke eigenvalues.

| | | |
|---|---|---|
| $u_1$ | $T_2$ | $(1-4X)(1-8X)(1+2X^2)$ |
| | $T_3$ | $(1-9X)(1-27X)(1+2X+3X^2)$ |
| | $T_5$ | $(1-25X)(1-125X)(1-3X+5X^2)$ |
| $u_2$ | $T_2$ | $(1-X)(1-2X)(1+32X^2)$ |
| | $T_3$ | $(1-X)(1-3X)(1+18X+243X^2)$ |
| | $T_5$ | $(1-X)(1-5X)(1-75X+3125X^2)$ |
| $u_3$ | $T_2$ | $(1-2X)(1-4X)(1+3X+8X^2)$ |
| | $T_3$ | $(1-3X)(1-9X)(1+5X+27X^2)$ |
| | $T_5$ | $(1-5X)(1-25X)(1+12X+125X^2)$ |

In these examples, the cohomology is completely accounted for by the Eisenstein summand of (A.2.8). In fact, let $\Gamma'_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$ be the usual Hecke congruence subgroup of matrices upper-triangular modulo $N$. Then the cohomology classes above actually come from classes in $H^1(\Gamma'_0(N))$, that is from holomorphic modular forms of level $N$.

For $N = 11$, the space of weight two cusp forms $S_2(11)$ is 1-dimensional. This cusp form $f$ lifts in two different ways to $H^5(\Gamma_0(11);\mathbb{C})$, which can be seen from the quadratic part of the Hecke polynomials for the $u_i$. Indeed, for $u_i$ the quadratic part is exactly the inverse of the local factor for the $L$-function attached to $f$, after the substitution $X = l^{-s}$. For $u_2$, we see that the lift is also twisted by the square of the cyclotomic character. (In fact the linear terms of the Hecke polynomials come from powers of the cyclotomic character.)

For $N = 19$, the space of weight two cusp forms $S_2(19)$ is again 1-dimensional. The classes $u_1$ and $u_2$ are lifts of this form, exactly as for $N = 11$. The class $u_3$, on the other hand, comes from $S_4(19)$, the space of weight 4 cusp forms on $\Gamma'_0(19)$. In fact, $\dim S_4(19) = 4$, with one Hecke eigenform defined over $\mathbb{Q}$ and another defined over a totally real cubic extension of $\mathbb{Q}$. Only the rational weight four eigenform contributes to $H^5(\Gamma_0(19);\mathbb{C})$. One can show that whether or not a weight four cuspidal eigenform $f$ contributes to the cohomology of $\Gamma_0(N)$ depends only on the sign of the functional equation of $L(f,s)$ [**Wes**]. This phenomenon is typical of what one encounters when studying Eisenstein cohomology.

In addition to the lifts of weight 2 and weight 4 cusp forms, for other levels one finds lifts of Eisenstein series of weights 2 and 4, and lifts of cuspidal cohomology classes from subgroups of $\mathrm{SL}_3(\mathbb{Z})$. For some levels one finds cuspidal classes that appear to be lifts from the group of symplectic similitudes GSp(4). More details can be found in [**AGM02, AGM06**].

**A.5.14.** Here are some notes on the reduction algorithm and its implementation:

- Some additional care must be taken when selecting reducing points for the submodular symbols of $\mathbf{u}$. In particular, in practice one should choose $w$ for $\mathbf{v}$ such that $\sum \|\mathbf{v}_i(w)\|$ is minimized. Similar remarks apply when choosing a subdivision of the crosspolytopes in §A.5.10.
- In practice, the reduction algorithm has *always* terminated with a reduced 1-sharbly cycle $\xi'$ homologous to $\xi$. However, at the moment we cannot prove that this will always happen.
- Experimentally, the efficiency of the reduction step appears to be comparable to that of Theorem A.22. In other words the depth of the "reduction tree" associated to a given 1-sharbly $\mathbf{u}$ seems to be bounded by a polynomial in $\log\log\|\mathbf{u}\|$. Hence computing the Hecke action using this algorithm is extremely efficient.

  On the other hand, computing Hecke operators on $\mathrm{SL}_4$ is still a much bigger computation—relative to the level—than on $\mathrm{SL}_2$ and $\mathrm{SL}_3$. For example, the size of the full retract $W$ modulo $\Gamma_0(p)$ is roughly $O(p^6)$, which grows rapidly with $p$. The portion of the retract corresponding to $H^5$ is much smaller, around $p^3/10$, but this still grows quite quickly. This makes computing with $p > 100$ out of reach at the moment.

  The number of Hecke cosets grows rapidly as well, e.g. the number of coset representatives of $T(l,2)$ is $l^4 + l^3 + 2l^2 + l + 1$. Hence it is only feasible to compute Hecke operators for small $l$; for large levels only $l = 2$ is possible.

  Here are some numbers to give an idea of the size of these computations. For level 73, the rank of $H^5$ is 20. There are 39504 cells of codimension one and 4128 top-dimensional cells in $W$ modulo $\Gamma_0(73)$. The computational techniques in [**AGM02**] used at this level (a Lanczos scheme over a large finite field) tend to produce sharbly cycles supported on almost all the cells. Computing

$T(2,1)$ requires a reduction tree of depth 1, and produces as many as 26 reduced 1-sharblies for each of the 15 nonreduced Hecke images. Thus one cycle produces a cycle supported on as many as 15406560 1-sharblies, all of which must be converted to an appropriate cell of $W$ modulo $\Gamma$. And this is just what needs to be done for *one* cycle; don't forget that the rank of $H^5$ is 20.

In practice the numbers are slightly better, since the reduction step produces fewer 1-sharblies on average, and since the support of the initial cycle has size less than 39504. Nevertheless the orders of magnitude are correct.

- Using lifts is a convenient way to encode the global $\Gamma$-identifications in the cycle $\xi$, since it means we don't have to maintain a big data structure keeping track of the identifications on $\partial\xi$. However, there is a certain expense in computing the Hermite normal form. This is balanced by the benefit that working with the data $\Phi$ associated to $\xi$ allows us to reduce the supporting 1-sharblies $\mathbf{u}$ *independently*. This means we can cheaply parallelize our computation: each 1-sharbly, encoded as a 4-tuple $(\mathbf{u}, n(\mathbf{u}), \{\mathbf{v}\}, \{M(\mathbf{v})\})$, can be handled by a separate computer. The results of all these individual computations can then be collated at the end, when producing a $W$-cocycle.

## A.6. Complements and Open Problems

**A.6.1.** We conclude this appendix by giving some complements and describing some possible directions for future work, both theoretical and computational. Since a full explanation of the material in this section would involve many more pages, we will be brief and will provide many references.

**A.6.2. Perfect quadratic forms over number fields and retracts.** Since Voronoĭ's pioneering work [**Vor08**], it has been the goal of many to extend his results from $\mathbb{Q}$ to a general algebraic number field $F$. Recently Coulangeon [**Cou01**], building on work of Icaza and Baeza [**Ica97, BI97**], has found a good notion of *perfection* for quadratic forms over number fields.([12]) One of the key ideas in [**Cou01**] is that the correct notion of equivalence between Humbert forms involves not only the action of $\mathrm{GL}_n(\mathscr{O}_F)$, where $\mathscr{O}_F$ is the ring of integers of $F$, but also the action of a certain continuous group $U$ related to the units $\mathscr{O}_F^\times$. One of Coulangeon's basic results is that there are finitely many equivalence classes of perfect Humbert forms modulo these actions.

[12] Such forms are called *Humbert forms* in the literature.

On the other hand, Ash's original construction of retracts [**Ash77**] introduces a geometric notion of perfection. Namely he generalizes the Voronoĭ polyhedron $\Pi$, and defines a quadratic form to be perfect if it naturally indexes a facet of $\Pi$. What is the connection between these two notions? Can one use Coulangeon's results to construct cell complexes to be used in cohomology computations? One tempting possibility is to try to use the group $U$ to collapse the Voronoĭ cells of [**Ash77**] into a cell decomposition of the symmetric space associated to $\mathrm{SL}_n(F)$.

**A.6.3. The modular complex.** In his study of multiple $\zeta$-values, Goncharov has recently defined the *modular complex* $M^*$ [**Gon97, Gon98**]. This is an $n$-step complex of $\mathrm{GL}_n(\mathbb{Z})$-modules closely related both to the properties of multiple polylogarithms evaluated at $\mu_N$, the $N$th roots of unity, and to the action of $G_{\mathbb{Q}}$ on $\pi_{1,N} = \pi_1^l(\mathbb{P}^1 \smallsetminus \{0, \infty, \mu_N\})$, the pro-$l$ completion of the algebraic fundamental group of $\mathbb{P}^1 \smallsetminus \{0, \infty, \mu_N\}$.

Remarkably, the modular complex is very closely related to the Voronoĭ decomposition $\mathscr{V}$. In fact, one can succinctly describe the modular complex by saying that it is the chain complex of the cells coming from the top-dimensional Voronoĭ cone of type $A_n$. This is all of the Voronoĭ decomposition for $n = 2, 3$, and Goncharov showed that the modular complex is quasi-isomorphic to the full Voronoĭ complex for $n = 4$. Hence there is a precise relationship among multiple polylogarithms, the Galois action on $\pi_{1,N}$, and the cohomology of level $N$ congruence subgroups of $\mathrm{SL}_n(\mathbb{Z})$.

The question then arises, how much of the cohomology of congruence subgroups is captured by the modular complex for all $n$? Table A.3.2 indicates that asymptotically very little of the Voronoĭ decomposition comes from the $A_n$ cone, but this says nothing about the cohomology. The first interesting case to consider is $n = 5$.

**A.6.4. Retracts for other groups.** The most general construction of retracts $W$ known [**Ash84**] applies only to *linear* symmetric spaces. The most familiar example of such a space is $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$; other examples are the symmetric spaces associated to $\mathrm{SL}_n$ over number fields and division algebras.

Now let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic group, and let $X = G/K$ be the associated symmetric space. What can one say about cell complexes that can be used to compute $H^*(\Gamma; \mathscr{M})$? The theorem of Borel–Serre mentioned in §A.3.3 implies the vanishing of $H^k(\Gamma; \mathscr{M})$ for $k > \nu := \dim X - q$, where $q$ is the $\mathbb{Q}$-*rank* of $\Gamma$. For example, for the split form of $\mathrm{SL}_n$, the $\mathbb{Q}$-rank is $n-1$. For the split symplectic group $\mathrm{Sp}_{2n}$, the $\mathbb{Q}$-rank is $n$. Moreover, this bound is sharp: there will be coefficient modules $\mathscr{M}$ for which $H^\nu(\Gamma; \mathscr{M}) \neq 0$. Hence any minimal cell complex used to compute the cohomology of $\Gamma$ should have dimension $\nu$.

Ideally one would like to see such a complex realized as a subspace of $X$, and would like to be able to treat all finite-index subgroups of $\Gamma$ simultaneously. This leads to the following question: is there a $\Gamma$-equivariant deformation retraction of $X$ onto a regular cell complex $W$ of dimension $\nu$?

For $\mathbf{G} = \mathrm{Sp}_4$, McConnell–MacPherson showed that the answer is yes. Their construction begins by realizing the symplectic symmetric space $X_{\mathrm{Sp}}$ as a subspace of the special linear symmetric space $X_{\mathrm{SL}}$. They then construct subsets of $X_{\mathrm{Sp}}$ by intersecting the Voronoĭ cells in $X_{\mathrm{SL}}$ with $X_{\mathrm{Sp}}$. Through explicit computations in coordinates they prove that these intersections are cells, and give a cell decomposition of $X_{\mathrm{Sp}}$. By taking an appropriate dual complex (as suggested by Figures A.3.2 and A.3.3, and as done in [**Ash77**]) they construct the desired cell complex $W$.

Other progress has been recently made by Bullock [**Bul00**], Bullock–Connell [**BC06**], and Yasaki [**Yas05b, Yas05a**] in the case of groups of $\mathbb{Q}$-rank 1. In particular, Yasaki uses the *tilings* of Saper [**Sap97**] to construct an explicit retract for the unitary group $\mathrm{SU}(2,1)$ over the Gaussian integers. His method also works for Hilbert modular groups, although further refinement may be needed to produce a regular cell complex. Can one generalize these techniques to construct retracts for groups of arbitrary $\mathbb{Q}$-rank? Is there an analogue of the Voronoĭ decomposition for these retracts (i.e. a dual cell decomposition of the symmetric space)? If so can one generalize ideas in §§A.4–A.5 and use it to compute the action of the Hecke operators on the cohomology?

**A.6.5. Deeper cohomology groups.** The algorithm in §A.5 computes the Hecke action on $H^{\nu-1}(\Gamma)$. For $n > 4$, this group no longer contains cuspidal cohomology classes. Can one generalize this algorithm to compute the Hecke action on deeper cohomology groups? The first practical case is $n = 5$. Here $\nu = 10$, and the highest degree in which cuspidal cohomology can live is 8. This case is also interesting since the cohomology of full level has been studied [**EVGS02**].

Here are some indications of what one can expect. The general strategy is the same: for a $k$-sharbly $\xi$ representing a class in $H^{\nu-k}(\Gamma)$, begin by $\Gamma$-equivariantly choosing reducing points for the nonunimodular submodular symbols of $\xi$. This data can be packaged into a new $k$-sharbly cycle as in §A.5.7ff, but the crosspolytopes must be replaced with *hypersimplices*. By definition, the hypersimplex $\Delta(n,k)$ is the convex hull in $\mathbb{R}^n$ of the points $\{\sum_{i\in I} e_i\}$, where $I$ ranges over all order $k$ subsets of $\{1,\dots,n\}$, and $e_1,\dots,e_n$ denotes the standard basis of $\mathbb{R}^n$.

The simplest example is $n = 2$, $k = 2$. From the point of view of cohomology, this is even less interesting than $n = 2$, $k = 1$, since now we're computing the Hecke action on $H^{-1}(\Gamma)$! Nevertheless, the geometry here illustrates what one can expect in general.

Each 2-sharbly in the support of $\xi$ can be written as $[v_1, v_2, v_3, v_4]$ and determines 6 submodular symbols, of the form $[v_i, v_j]$, $i \neq j$. Assume for simplicity that all these submodular symbols are nonunimodular. Let $w_{ij}$ be the reducing point for $[v_i, v_j]$. Then use the 10 points $v_i, w_{ij}$ to label the vertices of the hypersimplex $\Delta(5, 2)$ as in Figure A.6.1 (note that $\Delta(5, 2)$ is 4-dimensional).

The boundary of this hypersimplex gives the analogue of (A.5.4). Which 2-sharblies will appear in $\xi'$? The boundary $\partial\Delta(5, 2)$ is a union of 5 tetrahedra and 5 octahedra. The outer tetrahedron won't appear in $\xi'$, since that is the analogue of the left of (A.5.4). The four octahedra sharing a triangular face with the outer tetrahedron also won't appear, since they disappear when considering $\xi'$ modulo $\Gamma$. The remaining 4 tetrahedra and the central octahedron survive to $\xi'$, and constitute the right of the analogue of (A.5.4). Note that we must choose a simplicial subdivision of the central octahedron to write the result as a 2-sharbly cycle, and that this must be done with care since it introduces a new submodular symbol.

If some submodular symbols are unimodular, then again one must consider iterated cones on hypersimplices, just as in §A.5.10. The analogues of these steps become more complicated, since there are now many simplicial subdivisions of a hypersimplex.([13]) There is one final complication: in general we cannot use reduced $k$-sharblies alone to represent cohomology classes. Thus one must terminate the algorithm when $\|\xi\|$ is less than some predetermined bound.

**A.6.6. Other linear groups.** Let $F$ be a number field, and let $\mathbf{G} = \mathbf{R}_{F/\mathbb{Q}}(\mathrm{SL}_n)$ (Example A.2). Let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic subgroup. Can one compute the action of the Hecke operators on $H^*(\Gamma)$?

There are two completely different approaches to this problem. The first involves the generalization of the modular symbol method. One can define the analogue of the Sharbly complex, and can try to extend the techniques of §§A.4–A.5.

This technique has been extensively used when $F$ is *imaginary quadratic* and $n = 2$. We have $X = \mathrm{SL}_2(\mathbb{C})/\,\mathrm{SU}(2)$, which is isomorphic to three-dimensional hyperbolic space $\mathfrak{h}_3$. The arithmetic groups $\Gamma \subset \mathrm{SL}_2(\mathscr{O}_F)$ are known as *Bianchi groups*. The retracts and cohomology of these groups

[13] Indeed, computing all simplicial subdivisions of $\Delta(n, k)$ is a difficult problem in convex geometry.

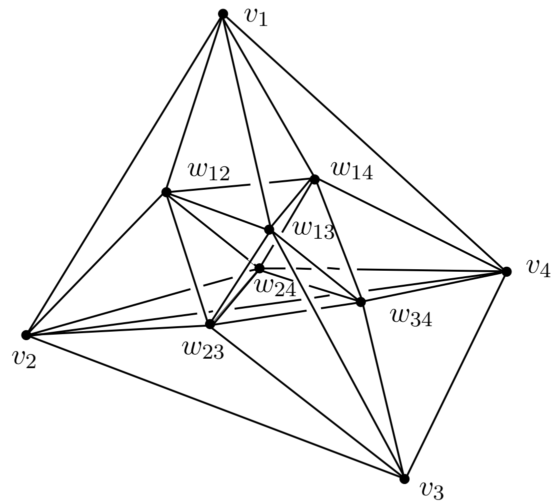



**Figure A.6.1.**

have been well-studied; as a representative sample of works we mention [**Men79, EGM98, Vog85, GS81**]

Such groups have $\mathbb{Q}$-rank 1, and thus have cohomological dimension 2. One can show that the cuspidal classes live in degrees 1 and 2. This means that we can use modular symbols to investigate the Hecke action on cuspidal cohomology. This was done by Cremona [**Cre84**] for *euclidean* fields $F$. In that case Theorem A.22 works with no trouble (the euclidean algorithm is needed to construct reducing points). For non-euclidean fields further work has been done by Whitley [**Whi90**], Cremona–Whitely [**CW94**] (both for principal ideal domains), Bygott [**Byg99**] (for $F = \mathbb{Q}(\sqrt{-5})$ and any field with class group an elementary abelian 2-group), and Lingham [**Lin05**] (any field with odd class number). Putting all these ideas together allows one to generalize the modular symbol method to *any* imaginary quadratic field [**Cre**].

For $F$ imaginary quadratic and $n > 2$, very little has been studied. The only related work to the best of our knowledge is that of Staffeldt [**Sta79**]. He determined the structure of the Voronoĭ polyhedron in detail for $\mathbf{R}_{F/\mathbb{Q}}(\mathrm{SL}_3)$, where $F = \mathbb{Q}(\sqrt{-1})$. We have $\dim X = 8$ and $\nu = 6$. The cuspidal cohomology appears in degrees $3, 4, 5$, so one could try to use the techniques of §A.5 to investigate it.

Similar remarks apply to $F$ *real quadratic* and $n = 2$. The symmetric space $X \simeq \mathfrak{h} \times \mathfrak{h}$ has dimension 4 and the $\mathbb{Q}$-rank is 1, which means $\nu = 3$. Unfortunately the cuspidal cohomology appears only in degree 2, which means modular symbols can't see it. On the other hand, 1-sharblies can see it, and so one can try to use ideas in §A.5 here to compute the Hecke operators. The data needed to build the retract $W$ already (essentially) appears in the literature for certain fields, see for example [**Ong86**].

The second approach shifts the emphasis from modular symbols and the sharbly complex to the Voronoĭ fan and its cones. For this approach we must assume that the group $\Gamma$ is associated to a *self-adjoint homogeneous cone* over $\mathbb{Q}$. (cf. [**Ash77**]). This class of groups includes arithmetic subgroups of $\mathbf{R}_{F/\mathbb{Q}}(\mathrm{SL}_n)$, where $F$ is a totally real or CM field. Such groups have all the nice structures in §A.3.2. For example, we have a cone $C$ with a $G$-action. We also have an analogue of the Voronoĭ polyhedron $\Pi$. There is a natural compactification $\tilde{C}$ of $C$ obtained by adjoining certain self-adjoint homogeneous cones of lower rank. The quotient $\Gamma\backslash\tilde{C}$ is singular in general, but can still be used to compute $H^*(\Gamma;\mathbb{C})$. The polyhedron $\Pi$ can be used to construct a fan $\mathscr{V}$ that gives a $\Gamma$-equivariant decomposition of all of $\tilde{C}$. But the most important structure we have is the Voronoĭ reduction algorithm: given any point $x \in \tilde{C}$, we can determine the unique Voronoĭ cone containing $x$.

Here is how this setup can be used to compute the Hecke action. Full details are in [**Gun99, GM03**]. We define two chain complexes $\mathbf{C}_*^V$ and $\mathbf{C}_*^R$. The latter is essentially the chain complex generated by all simplicial rational polyhedral cones in $\tilde{C}$; the former is the subcomplex generated by the Voronoĭ cones. These are the analogues of the sharbly complex and the chain complex associated to the retract $W$, and one can show that either can be used to compute $H^*(\Gamma;\mathbb{C})$. Take a cycle $\xi \in \mathbf{C}_*^V$ representing a cohomology class in $H^*(\Gamma;\mathbb{C})$ and act on it by a Hecke operator $T$. We have $T(\xi) \in \mathbf{C}_*^R$, and we must push $T(\xi)$ back to $\mathbf{C}_*^V$.

To do this, we use the linear structure on $\tilde{C}$ to subdivide $T(\xi)$ very finely into a chain $\xi'$. For each 1-cone $\tau$ in $\operatorname{supp}\xi'$, we choose a 1-cone $\rho_\tau \in \tilde{C} \smallsetminus C$ and assemble them using the combinatorics of $\xi'$ into a polyhedral chain $\xi''$ homologous to $\xi'$. Under certain conditions involved in the construction of $\xi'$, this chain $\xi''$ will lie in $\mathbf{C}_*^V$.

We illustrate this process for the split group $\mathrm{SL}_2$; more details can be found in [**Gun99**]. We work modulo homotheties, so that the three-dimensional cone $\tilde{C}$ becomes the extended upper halfplane $\mathfrak{h}^* := \mathfrak{h}\cup\mathbb{Q}\cup\{\infty\}$, with $\partial\tilde{C}$ passing to the cusps $\mathfrak{h}^* \smallsetminus \mathfrak{h}$. As usual top-dimensional Voronoĭ cones become the triangles of the Farey tessellation, and the cones $\rho_\tau$ become cusps. Given any $x \in \mathfrak{h}$, let $R(x)$ be the set of cusps of the unique triangle or edge containing $x$ (this can be computed using the Voronoĭ reduction algorithm). Extend $R$ to a function on $\mathfrak{h}^*$ by putting $R(u) = \{u\}$ for any cusp $u$.

In $\mathfrak{h}$, the support of $T(\xi)$ becomes a geodesic $\mu$ between two cusps $u$, $u'$, in other words the support of a modular symbol $[u, u']$ (Figure A.6.2). Subdivide $\mu$ by choosing points $x_0, \dots, x_n$ such that $x_0 = u$, $x_n = u'$, and $R(x_i) \cap R(x_{i+1}) \neq \varnothing$. (This is easily done, for example by repeatedly barycentrically subdividing $\mu$). For each $i < n$ choose a cusp $q_i \in R(x_i) \cap R(x_{i+1})$, and put $q_n = u'$. Then we have a relation in $H^1$

$$[u, u'] = [q_0, q_1] + \cdots + [q_{n-1}, q_n]. \tag{A.6.1}$$

Moreover, each $[q_i, q_{i+1}]$ is unimodular, since $q_i$ and $q_{i+1}$ are both vertices of a triangle containing $x_{i+1}$. Upon lifting (A.6.1) back to $\mathbf{C}_*^R$, the cusps $q_i$ become the 1-cones $\rho_\tau$, and give us a relation $T(\xi) = \xi'' \in \mathbf{C}_*^V$.

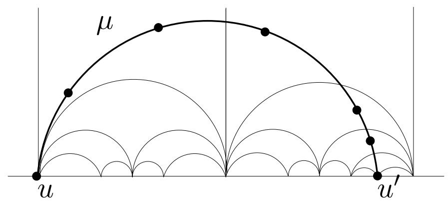



**Figure A.6.2.** A subdivision of $\mu$; the solid dots are the $x_i$. Since the $x_i$ lie in the same or adjacent Voronoĭ cells, we can assign cusps to them to construct a homology to a cycle in $\mathbf{C}_*^V$.

**A.6.7. The sharbly complex for general groups.** In [**Gun00b**] we generalized Theorem A.22 (without the complexity statement) to the symplectic group $\mathrm{Sp}_{2n}$. Using this algorithm and the symplectic retract [**MM93, MM89**] one can compute the action of the Hecke operators on the top degree cohomology of subgroups of $\mathrm{Sp}_4(\mathbb{Z})$.

More recently, Toth has investigated modular symbols for other groups. He showed that the unimodular symbols generate the top degree cohomology groups for $\Gamma$ an arithmetic subgroup of a split classical group or a split group of type $E_6$ or $E_7$ [**Tot05**]. His technique of proof is completely different from that of [**Gun00b**]. In particular he doesn't give an analogue of the Manin trick. Can one extract an algorithm from Toth's proof that can be used to explicitly compute the action of the Hecke operators on cohomology?

The proof of the main result of [**Gun00b**] uses a description of the relations among the modular symbols. These relations were motivated by the structure of the cell complex in [**MM93, MM89**]. The modular symbols and these relations are analogues of the groups $S_0$ and $S_1$ in the sharbly complex. Can one extend these combinatorial constructions to form a *symplectic sharbly complex*? What about for general groups $\mathbf{G}$?

Already for $\mathrm{Sp}_4$ resolution of this question would have immediate arithmetic applications. Indeed, Harder has a beautiful conjecture about certain congruences between holomorphic modular forms and Siegel modular forms of full level [**Hara**]. Examples of these congruences were checked numerically in [**Hara**] using techniques of [**FvdG**] to compute the Hecke action.

However to investigate higher levels one needs a different technique. The relevant cohomology classes live in $H^{\nu-1}(\Gamma;\mathscr{M})$, so one only needs to understand the first three terms of the complex $S_0 \leftarrow S_1 \leftarrow S_2$. We understand $S_0$, $S_1$ from [**Gun00b**]; the key is understanding $S_2$, which should encode relations among elements of $S_1$. If one could do this, and then could generalize the techniques of [**Gun00a**], one would have a way to investigate Harder's conjecture.

**A.6.8. Generalized modular symbols.** We conclude this appendix by discussing a geometric approach to modular symbols. This complements the algebraic approaches presented in this book, and leads to many new interesting phenomena and problems.

Suppose $\mathbf{H}$ and $\mathbf{G}$ are connected semisimple algebraic groups over $\mathbb{Q}$ with an injective map $f\colon \mathbf{H} \to \mathbf{G}$. Let $K_H$ be a maximal compact subgroup of $H = \mathbf{H}(\mathbb{R})$, and suppose $K \subset G$ is a maximal compact subgroup containing $f(K_H)$. Let $X = G/K$ and $Y = H/K_H$.

Now let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Let $\Gamma_H = f^{-1}(\Gamma)$. We get a map $\Gamma_H\backslash Y \to \Gamma\backslash X$, and we denote the image by $S(H,\Gamma)$. Any compactly supported cohomology class $\xi \in H_c^{\dim Y}(\Gamma\backslash X;\mathbb{C})$ can be pulled back via $f$ to $\Gamma_H\backslash Y$ and integrated to obtain a complex number. Hence $S(H,\Gamma)$ defines a linear form on $H_c^{\dim Y}(\Gamma\backslash X;\mathbb{C})$. By Poincaré duality, this linear form determines a class $[S(H,\Gamma)] \in H^{\dim X-\dim Y}(\Gamma\backslash X;\mathbb{C})$, called a *generalized modular symbol*. Such classes have been considered by many authors, for example [**AB90, SV03, Har05, AGR93**]

As an example, we can take $\mathbf{G}$ to be the split form of $\mathrm{SL}_2$, and can take $f\colon \mathbf{H} \to \mathbf{G}$ to be the inclusion of connected component of the diagonal subgroup. Hence $H \simeq \mathbb{R}_{>0}$. In this case $K_H$ is trivial. The image of $Y$ in $X$ is the ideal geodesic from 0 to $\infty$. One way to vary $f$ is by taking an $\mathrm{SL}_2(\mathbb{Q})$-translate of this geodesic, which gives a geodesic between two cusps. Hence we can obtain the support of any modular symbol this way. This example generalizes to $\mathrm{SL}_n$ to yield the modular symbols in §A.4. Here $H \simeq (\mathbb{R} > 0)^{n-1}$. Note that $\dim Y = n-1$, so the cohomology classes we're constructed live in the top degree $H^{\nu}(\Gamma\backslash X;\mathbb{C})$.

Another family of examples is provided by taking $\mathbf{H}$ to be a Levi factor of a parabolic subgroup; these are the modular symbols studied in [**AB90**].

There are many natural questions to study for such objects. Here are two:

- Under what conditions on $\mathbf{G}, \mathbf{H}, \Gamma$ is $[S(H,\Gamma)]$ nonzero? This question is connected to relations between periods of automorphic forms and functoriality lifting. There are a variety of partial results known, see for example [**SV03, AGR93**].
- We know the usual modular symbols span the top-degree cohomology for any arithmetic group $\Gamma$. Fix a class of generalized modular symbols by fixing the pair $\mathbf{G}, \mathbf{H}$ and fixing some class of maps $f$. How much of the cohomology can one span for a general arithmetic group $\Gamma \subset \mathbf{G}(\mathbb{Q})$?

  A simple example is given by the Ash–Borel construction for $\mathbf{G} = \mathrm{SL}_3$ and $\mathbf{H}$ a Levi factor of a rational parabolic subgroup $\mathbf{P}$ of type $(2,1)$. In this case $H \simeq \mathrm{SL}_2(\mathbb{R}) \times \mathbb{R}_{>0}$, and sits inside $G$ via

$$g \begin{pmatrix} \alpha^{-1}M & 0 \\ 0 & \alpha \end{pmatrix} g^{-1}, \quad M \in \mathrm{SL}_2(\mathbb{R}), \quad \alpha \in \mathbb{R}_{>0}, \quad g \in \mathrm{SL}_3(\mathbb{Q}).$$

  For $\Gamma \subset \mathrm{SL}_3(\mathbb{Z})$ these symbols define a subspace

$$S_{(2,1)} \subset H^2(\Gamma \backslash X; \mathbb{C}).$$

  Are there $\Gamma$ for which $S_{(2,1)}$ equals the full cohomology space? For general $\Gamma$ how much is captured? Is there a nice combinatorial way to write down the relations among these classes? Can one cook up a generalization of Theorem A.22 for these classes and use it to compute Hecke eigenvalues?

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