# Units, polyhedra, and a conjecture of Satake

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## Abstract

Let $F/\mathbf{Q}$ be a totally real number field of degree $n$. We explicitly evaluate a certain sum of rational functions over a infinite fan of $F$-rational polyhedral cones in terms of the norm map $\mathrm{N}\colon F \to \mathbf{Q}$. This is an important step in Sczech's program to prove a conjecture of Satake, which connects special values of $L$-series associated to cusp singularities with intersection numbers of divisors in their toroidal resolutions [Sc2].


## 1. Introduction

**1.1.** Let $F/\mathbf{Q}$ be a totally real number field of degree $n$ with ring of integers $\mathbf{Z}_F$. Let $M \subset F$ be a $\mathbf{Z}$-module of rank $n$, and let $\rho \in F$ determine a coset $M + \rho$. Let $d(M)$ be the square root of the discriminant of $M$. Recall that $x \in F$ is said to be *totally positive* if $v(x) > 0$ for each infinite place $v$ of $F$. Let $U \subset \mathbf{Z}_F^\times$ be the totally positive units, and let $U_{M+\rho} \subset U$ be the subgroup with $\varepsilon(M + \rho) = M + \rho$. Choose a finite index subgroup $V \subset U_{M+\rho}$. Then the pair $(M + \rho, V)$ determines a collection of special values

$$L(M + \rho, V; s) := \sum_{\mu \in (M+\rho)/V} \mathrm{N}(\mu)^{-s}, \quad s = 1, 2, 3, \ldots.$$

Here $\mathrm{N}\colon F \to \mathbf{Q}$ is the norm map, and in the sum we omit $\mu = 0$. These series converge absolutely if $s > 1$, but only conditionally if $s = 1$. The theorem of Klingen and Siegel asserts that these special values are—up to $d(M)$ and powers of $\pi$—cyclotomic numbers, and moreover rational if $\rho \in M$. For example, let $F = \mathbf{Q}(\sqrt{3})$, $M = \mathbf{Z}1 + \mathbf{Z}(\sqrt{3}/3)$, $\rho = 0$, and $V = U_M = U$. Then $d(M) = 2/\sqrt{3}$, and by computing the $L$-series numerically to high accuracy (using GP-Pari [GP]) we easily find

$$L(M, V; 1) = -\frac{\pi^2\sqrt{3}}{6}, \quad L(M, V; 2) = \frac{\pi^4\sqrt{3}}{6}, \quad L(M, V; 3) = -\frac{\pi^6\sqrt{3}}{36}.$$

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In [Sa], Satake proposed a geometric interpretation of these special values in terms of certain intersection numbers. The data $(M, V, s)$ determines a cusp singularity $Y^*(M, V, s)$. This is a complex analytic space of complex dimension $ns$ with an isolated singular point (the cusp), and is obtained by (partially) compactifying the quotient of a bounded symmetric domain by an arithmetic subgroup. Examples of this class of singularities include the cusps in the Baily-Borel compactifications of Hilbert and Picard modular varieties. Using toroidal geometry [AMRT, KKMS], one can construct a desingularization $\tilde{Y}(M, V, s) \to Y^*(M, V, s)$ with an exceptional divisor with finitely many irreducible components $\{D_\tau\}$; for $s = 1$ these components are toric varieties, and for $s > 1$ they are toric variety bundles over an $n(s-1)$-dimensional abelian variety $A$. This abelian variety depends on additional data used to construct $\tilde{Y}(M, V, s)$, but the exact nature of this data is not important for us. Then Satake's conjecture, in the special case $\rho \in M$, becomes

$$\frac{\mathrm{Vol}(A)d(M)((s-1)!)^n}{(2\pi i)^{ns}} L(M, V; s) = \frac{1}{(ns)!} \Big(\sum_\tau B_\tau D_\tau\Big)^{ns}. \tag{S?}$$

Here $B_\tau$ is a "Bernoulli symbol" defined such that $B_\tau^k$ is the $k$th Bernoulli number $B_k$, independent of $\tau$. The modification to $\rho \notin M$ is slightly more complicated to state, but the geometry remains unchanged. Continuing the example from above, we find that the resolutions $\tilde{Y}(M, V; s)$ have exceptional divisors $D_s$ with two irreducible components $D_{s,1}, D_{s,2}$. Computing the relevant intersection numbers yields

$$\kappa D_{1,1}^2 = -2, \quad \kappa D_{1,2}^2 = -3, \quad \kappa D_{1,1}D_{1,2} = 2,$$

$$\kappa D_{2,1}^4 = 0, \quad \kappa D_{2,2}^4 = 0, \quad \kappa D_{2,1}^2 D_{2,2}^2 = 3,$$

$$\kappa D_{3,1}^6 = -12, \quad \kappa D_{3,2}^6 = -81/2, \quad \kappa D_{3,1}^4 D_{3,2}^2 = -18, \quad \kappa D_{3,1}^2 D_{3,2}^4 = -27,$$

where $\kappa$ denotes the inverse volume of the appropriate abelian variety. With these intersection numbers, (S) becomes

$$\begin{aligned}
L(M, V; 1) &= -\frac{4\pi^2}{2!}\frac{\sqrt{3}}{2}\Big(-3B_2 + 4\big(B_1\big)^2 - 2B_2\Big) = -\frac{\pi^2\sqrt{3}}{6},\\
L(M, V; 2) &= \frac{16\pi^4}{4!}\frac{\sqrt{3}}{2}\Big(0 \cdot B_4 + 6 \cdot 3\big(B_2\big)^2 + 0 \cdot B_4\Big) = \frac{\pi^2\sqrt{3}}{6},\\
L(M, V; 3) &= -\frac{64\pi^6}{6!(2!)^2}\frac{\sqrt{3}}{2}\Big(-12B_6 - 15 \cdot 18 B_4 B_2 - 15 \cdot 27 B_2 B_4 - \frac{81}{2}B_6\Big) = -\frac{\pi^6\sqrt{3}}{36},
\end{aligned}$$

where we use the Bernoulli numbers $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, and where we omit $B_0 = 1$.

**1.2.** Proofs of (S?) have appeared in the literature in special cases, and always for $\rho \in M$. For $s = 1$ and $F$ real quadratic, the conjecture for the $L$-value $L(M, V; 1)$ was proved by Hirzebruch [Hir], as a by-product of his resolution of the cusp singularities of Hilbert modular surfaces. For all totally real fields and $s = 1$, independent proofs were given by Atiyah-Donnelly-Singer [ADS] and Müller [Mül].

The most general results are due to Ogata [Oga], who verified (S?) for all $n$ and odd $s$. The restriction to odd $s$ arises as follows. Like [ADS] and [Mül], Ogata considers the $L$-function

$$L^{**}(M,V;s) := \sum_{\mu\in M/V} \operatorname{sign}(\mathrm{N}(\mu))|\mathrm{N}(\mu)|^{-s},$$

which converges in a suitable halfplane, has an analytic continuation to the entire complex plane, and satisfies a functional equation of the form $s \leftrightarrow 1-s$. He then studies special values of $L^{**}$ at nonpositive integral $s$. The special values of $L^{**}$ coincide with those of $L$ for odd positive integral $s$, but not for even positive integral $s$—in fact for even positive $s$ the series $L^{**}$ has a much more complicated special value (analogous to the special values of the Riemann zeta function at odd integers $\geq 3$), to which the theorem of Klingen and Siegel doesn't apply.

**1.3.** The proofs of [ADS, Mül, Oga] rely on the Atiyah-Patodi-Singer index theorem, applied to the the manifold with boundary obtained by "cutting off" the cusp singularity. In [Sc2] Sczech takes a different tack, and describes a program to prove Satake's conjecture that avoids the difficult analysis of the index theorem. Instead, Sczech shows how the special values are built from the intersection numbers of the cusp divisors.

As a first step, one unfolds the $L$-series, in the style of Hecke, from a sum over $(M+\rho)/V$ to a sum over the full coset $M+\rho$. This is formally done by writing

$$L(M+\rho,V;s) \;=\; \sum_{\mu\in(M+\rho)/V} \frac{1}{\mathrm{N}(\mu)^s} \;=\; \sum_{\mu\in(M+\rho)/V}\sum_{\sigma\in C} f_s(\sigma,\mu), \tag{$\bullet$}$$

where $C$ is a certain $V$ invariant $F$-*rational polyhedral fan* supported in $(\mathbf{R}_{>0})^n$, and the function $f_s(\sigma,x)$ is, for each fixed positive integer $s$ and each $\sigma\in C$, a certain rational function in $n$ variables which Sczech constructs explicitly from the data defining the fan.

Next, the $V$-equivariance properties of the rational functions $f_s$ allow one to write the outer sum on the right of $(\bullet)$ as a sum over $M+\rho$ and the inner sum as a sum over $C/V$. Since $C$ is finite modulo $V$, this shows that the $L$-value can be rewritten as a finite sum of "full sums," where a full sum is a sum over the affine lattice $M+\rho$ of certain rational functions. Each full sum is equal to an Eisenstein series that can be explicitly evaluated using the formulas derived in [Sc1] as higher-dimensional Dedekind sums. On the intersection number side, the components of the divisor of the resolution can be indexed by the top-dimensional cones in $C/V$, and the intersection numbers can be computed by applying differential operators to the $f_s$.

**1.4.** The main purpose of this paper is to provide a proof of the inner sum in $(\bullet)$ in a special case. Namely, we prove the identity

$$\frac{1}{\mathrm{N}(\mu)^s} = \sum_{\sigma\in C} f_s(\sigma,\mu) \tag{$\star$}$$

under the assumptions that $s = 1$, the sum is taken over the top-dimensional cones, and $\mu$ is *totally positive*. This is not quite enough to complete the arguments in [Sc2] and to give a proof of (S?), but it is a key step. The restrictions to $s = 1$ and the top-dimensional cones are harmless, and pose no obstruction to completing [Sc2]. On the other hand assuming $\mu$ totally positive is essential for our arguments, and at present we do not know how to eliminate this condition.

We remark that the convergence of ($\star$) is delicate. Since the sum does not have a natural ordering, it has a meaningful value only if it converges absolutely. In [Sc2] the absolute convergence of the sum is established for all $x_0 \in F \subseteq \mathbf{R}^n$ (here we regard $F$ as a dense subset of $\mathbf{R}^n$ via its $n$ embeddings into $\mathbf{R}$). One might naturally ask if the sum converges absolutely for all $x_0 \in \mathbf{R}^n$, but one can show this is not the case: there is a dense subset $\Sigma \subseteq \mathbf{R}^n$ such that the sum in ($\star$) fails to converge absolutely for all $x_0 \in \Sigma$.

For the rest of this introduction we sketch the proof of Theorem 1 and give a guide to the contents of our article. For $s = 1$ and $\sigma$ a top-dimensional cone $t$, the rational function $f_s(\sigma, x)$ becomes a function that we denote $h^*(t)(x)$. Hence we must show

$$\sum_t h^*(t)(x_0) = \frac{1}{\mathrm{N}(x_0)}, \tag{$\star_1$}$$

where the sum is taken over all top-dimensional cones $t \in C$, and where $x_0$ is totally positive.

The rational function $h^*(t)(x)$ satisfies a certain signed additivity property with respect to simplicial subdivisions of top-dimensional cones (Proposition 1), and $h^*(\Delta_\infty)(x_0)$ is exactly equal to $1/\mathrm{N}(x_0)$. Hence a natural idea to prove ($\star_1$) is to construct a sequence of partial sums of rational functions $\{S_N(x)\}$ by constructing a finite sequence of unions of top-dimensional cones $\{\Sigma_N\}$ such that $\Sigma_N \to \Delta_\infty$, and then by defining $S_N(x)$ by summing $h^*$ over the cones in $\Sigma_N$:

$$S_N(x) = \sum_{t \in \Sigma_N} h^*(t)(x).$$

One can then try to show that the sequence $S_N(x_0) \to h^*(\Delta_\infty)(x_0)$ as $N \to \infty$ by applying the additivity relation.

The main difficulty in carrying this out is that the value of the partial sum $S_N(x_0)$ depends rather subtlely on the geometric properties of $\Sigma_N$. For example, the singular hyperplanes of the rational function $S_N(x)$ are the linear spans of the cones in the boundary $\partial\Sigma_N$ of $\Sigma_N$, and so one must construct $\Sigma_N$ so that these linear spans are far from $x_0$ for large $N$. Furthermore, if $\partial\Sigma_N$ is complicated there is highly nontrivial cancellation among the terms in $S_N(x_0)$, which makes a direct estimate of the error term infeasible.

To resolve these difficulties, we first define a notion of *cycles* built from rational polyhedral cones, and show that $h^*$ induces a cocycle with coefficients in the field of rational functions in $n$ variables (§§3–7). This allows us to work with "formal polyhedra" instead of actual convex polyhedra, and to avoid explicitly dealing with cancellations. Then we analyze the action of the unit group $V$ on $\Delta_\infty$ to construct a sequence of convex polyhedral cones $\{\Sigma_N\}$ exhausting $\Delta_\infty$, and such that the singular hyperplanes spanned by $\partial\Sigma_N$ are far from $x_0$ (§8). Finally we apply a geometric interpretation of $h^*$ due to Hurwitz [Hur] to complete the proof of Theorem 1 (§9).

**1.5.** We thank Robert Sczech for suggesting this problem to us, for giving us access to [Sc2], and for many interesting conversations. We thank the NSF for support, and the first named author is grateful for the support of the Max Planck Institut in Bonn.

## 2. Notation and statement of the main result

**2.1.** We retain the notation from §1. In particular, $F/\mathbf{Q}$ is a totally real number field of degree $n$ with ring of integers $\mathbf{Z}_F$, $M \subset F$ is a rank $n$ $\mathbf{Z}$-module, $\rho \in F$ gives a coset $M+\rho$, and $V \subset U_{M+\rho} \subset U$ is a finite-index subgroup of the totally positive units preserving $M+\rho$.

We fix an ordering $v_1, \ldots, v_n$ of the infinite places of $F$, and identify $F$ with its image in $\mathbf{R}^n$ given by these embeddings. We use the notation $x^{(i)}$ for $v_i(x)$. Let $\langle\ ,\ \rangle \colon (\mathbf{R}^n)^2 \to \mathbf{R}$ be the standard scalar product. Note that if $x, y \in F$, then $\langle x, y\rangle = \mathrm{Tr}(xy)$. We will use this scalar product to identify $\mathbf{R}^n$ with its dual.

**2.2.** We recall some notations about cones and fans. For more information we refer to [Ful]. A subset $\sigma$ of a real vector space $W$ is called a *cone* if

(1) $\sigma \cap -\sigma = \{0\}$, and

(2) $x \in \sigma$ implies $\lambda x \in \sigma$ for all $\lambda \in \mathbf{R}_{\geq 0}$.

For any cone $\sigma$, the *dual cone* $\sigma^*$ is defined by

$$\sigma^* := \bigl\{x \bigm| \langle x, y\rangle \geq 0 \quad \text{for all } y \in \sigma\bigr\}.$$

A cone is *polyhedral* if it is the convex hull of finitely many half-lines, *simplicial* if the number of these lines can be taken to be the dimension of the linear span $\mathbf{R}\sigma$ of $\sigma$, and *rational* if $W$ has a $\mathbf{Q}$-structure and these half-lines each contain nonzero points in $W(\mathbf{Q})$. Any rational polyhedral cone $\sigma$ contains a collection of subcones, the *faces* of $\sigma$. We write $\tau < \sigma$ to indicate that $\tau$ is a face of $\sigma$. Faces of codimension 1 are called *facets*.

**2.3.** Let $C$ be a set of rational polyhedral cones. Then $C$ is a *fan* if

(1) $\sigma \in C$ and $\tau < \sigma$ implies $\tau \in C$, and

(2) if $\sigma, \tau \in C$, then $\sigma \cap \tau$ is a face of each.

We write $C(k)$ for the subset of $C$ consisting of $k$-dimensional cones. Following [Sc2], we reserve the notation $t$ (respectively $\tau$) for an element of $C(n)$ (resp. $C(1)$).

Given a cone $\sigma \in C$, the *star* of $\sigma$ is the set

$$\mathrm{St}(\sigma) := \left\{ \sigma' \in C \;\middle|\; \sigma < \sigma' \right\},$$

and the *link* of $\sigma$ is the set

$$\mathrm{Lk}(\sigma) := \left\{ \sigma' \;\middle|\; \sigma' < t \quad \text{for some } t \in \mathrm{St}(\sigma), \text{ and } \sigma \not< \sigma' \right\}.$$

A fan $C'$ is a *refinement* of $C$ if every cone in $C$ can be written as a union of cones in $C'$. Given any two rational polyhedral fans, there exists a rational fan that is a common refinement of each.

Recall that $\Delta_\infty$ is the totally positive chamber $(\mathbf{R}_{>0})^n \subset \mathbf{R}^n$. Throughout this paper, we will only consider polyhedral fans $C$ satisfying the following properties:

(1) $C$ is locally finite, apart from $\{0\} \in C$,

(2) $C$ is simplicial,

(3) $C$ is $F$-rational, i.e. each $\sigma \in C$ is generated by half-lines defined over $F$,

(4) there is a finite-index subgroup $V$ of the totally positive units that acts on $C$ via the embedding of $F$ into $\mathbf{R}^n$, and

(5) $C$ gives a decomposition of the totally positive chamber:

$$\bigcup_{\substack{\sigma \in C \\ \sigma \neq \{0\}}} \sigma = \Delta_\infty.$$

Note that this last condition implies $|C|$ is infinite. We call fans satisfying conditions (1)–(5) *good fans*. The existence of good fans was proved in great generality by Ash in Chapter II of [AMRT] (there, a good fan is called a $\Gamma$-*admissible decomposition*).

**2.4.** In what follows, we use the notational convention of multivariables. In particular, let $x$ be a real multivariable with components $x_1, \ldots, x_n$, and let $A = (A_1, \ldots, A_n)$ be an $n$-tuple of vectors in $\mathbf{R}^n$. Let $\mathbf{C}(x)$ be the field of rational functions in the variables $x_1, \ldots, x_n$. Our goal is to define two maps

$$\begin{aligned} h, h^*\colon (\mathbf{R}^n)^n &\longrightarrow \mathbf{C}(x) \\ A &\longmapsto h(A)(x), h^*(A)(x) \end{aligned}$$

that will play an important role in the sequel.

First, we define the function $h(A)(x) \in \mathbf{C}(x)$ by

$$h(A)(x) = \frac{\det A}{\langle x, A_1\rangle \cdots \langle x, A_n\rangle}.$$

Next we define the "dual function" $h^*(A)(x)$ as follows. If the set $A$ is linearly dependent we put $h^*(A)(x) = 0$. Otherwise we let $B = (B_1, \dots, B_n)$ be the dual basis to $A$ with respect to our inner product, and put

$$h^*(A)(x) := h(B)(x).$$

As a function of the $A_i$, both $h$ and $h^*$ are homogeneous of degree 0. Moreover, if $E = (e_1, \dots, e_n)$ is the canonical basis of $\mathbf{R}^n$, then $h(E)(x) = h^*(E)(x) = 1/\mathrm{N}(x)$, where for $x \in \mathbf{R}^n$ we put $\mathrm{N}(x) = \prod x_i$ (this agrees with the usual norm map on $F \subset \mathbf{R}^n$). Each of these function also enjoys a cocycle property:

**Proposition 1.** [Sc2] *Given $n+1$ vectors $A_0, \dots, A_n$, let $A^{(i)}$ be the tuple*

$$(A_0, \dots, \hat{A}_i, \dots, A_n),$$

*where the hat means to delete the ith component. Then we have*

$$\sum_{i=0}^{n} (-1)^i h(A^{(i)})(x) = \sum_{i=0}^{n} (-1)^i h^*(A^{(i)})(x) = 0.$$

**2.5.** We recall the geometric interpretation of $h$ due to Hurwitz [Hur]. Suppose $\langle x, A_i\rangle \neq 0$ for $A_1, \dots, A_n$, and $\det A \neq 0$. The points $A_i$ determine $2^n$ simplicial cones in $\mathbf{R}^n$, and hence $2^{n-1}$ regions in $\mathbf{P}^{n-1} = \mathbf{P}(\mathbf{R}^n)$. Among these regions is a unique region $R$ that misses the hyperplane $\{y \mid \langle x, y\rangle = 0\}$. Then up to sign, the value of $h(A)(x)$ is given by the integral

$$\int_R \Omega_x, \quad \text{where} \quad \Omega_x = \frac{(n-1)!}{\langle x, y\rangle^n} \sum_{i=1}^{n} (-1)^{i-1} y_i \, dy_1 \cdots \hat{dy_i} \cdots dy_n.$$

Here $\int_R \Omega_x$ is equal to the euclidean area of the region $R$ viewed as a subset of $\mathbf{R}^n$ as follows: $R \subseteq \mathbf{P}^{n-1} \backslash x^{\perp} = \mathbf{R}^n$. Moreover, the sign is determined by fixing an orientation on an affine chart containing $R$.

**2.6.** We extend the notation for $h$ and $h^*$ as follows. Recall that $M \subset \mathbf{R}^n$ is a lattice. Given any 1-cone $\tau \in C(1)$, let $A_\tau \in \tau \cap M$ be the nonzero point closest to the origin. Fix a global orientation on $\mathbf{R}^n$. Then if $t \in C(n)$, we let $A_t$ be the $n$-tuple $(A_\tau \mid \tau < t)$, where the points are positively ordered with respect to the orientation. Then we define

$$h(t)(x) := h(A_t)(x),$$

and define $h^*(t)(x)$ by using the dual basis to $A_t$. Note that the singular hyperplanes of $h^*(t)(x)$ are exactly the linear spans of the facets of $t$.

**2.7.** Now consider the infinite sum of rational functions

$$S(C)(x) := \sum_{t\in C(n)} h^*(t)(x).$$

We want to evaluate this sum at $x = x_0 \in \Delta_\infty$, but we must be careful because singular hyperplanes of some terms may pass through $x_0$. Let $\mathcal{U}_{x_0}$ be the set of singular cones determined by $x_0$:

$$\mathcal{U}_{x_0} := \{\sigma \in C \mid \dim(\sigma) < n, x_0 \in \mathbf{R}\sigma, x_0 \notin \mathbf{R}\sigma' \quad \text{for all} \quad \sigma' < \sigma\}.$$

If $\mathcal{U}_{x_0} = \emptyset$, then every term in $S(C)(x_0)$ is well defined, and we say that $x_0$ is *nonsingular* with respect to $C$. One can show in this case that if $x_0 \in F$ then $S(C)(x_0)$ is absolutely convergent [Sc2].

On the other hand, if $\mathcal{U}_{x_0} \neq \emptyset$, then we must make the following modifications to $S(C)$. For any $\sigma \in \mathcal{U}_{x_0}$, in $S(C)(x)$ we replace the sum of set of rational functions

$$H_\sigma = \{h^*(t)(x) \mid t \in \mathrm{St}(\sigma) \cap C(n)\}$$

with the finite sum $\Theta_\sigma := \sum_{h\in H_\sigma} h$. We do this for all $\sigma \in \mathcal{U}_{x_0}$, and add together the resulting rational functions.

**Proposition 2.** *All terms in the sum constructed by the above procedure are well-defined.*

*Proof.* Let $\sigma \in \mathcal{U}_{x_0}$. We must show that $x_0$ does not lie in the singular hyperplanes of the function $\Theta_\sigma$. By Proposition 1, the singular hyperplanes of $\Theta_\sigma$ are the linear spans of the dimension $n-1$ cones in $\mathrm{Lk}(\sigma)$. Thus it suffices to show that $\mathrm{Lk}(\sigma) \cap \mathcal{U}_{x_0} = \emptyset$.

Suppose not and let $\sigma'$ be a member of this intersection. By assumption, $x_0 \in \mathbf{R}\sigma' \cap \mathbf{R}\sigma = \mathbf{R}(\sigma' \cap \sigma)$. Since $\sigma' \not> \sigma$, the minimality of $\sigma$ is contradicted. ∎

One can also show in this case that if $x_0 \in F$, the resulting series is again absolutely convergent [Sc2].

**Lemma 1.** *Let $x_0 \in F$, and let $C$ be a good fan. If $C'$ is any other good fan, then $S(C)(x_0) = S(C')(x_0)$.*

*Proof.* First suppose that $C'$ is a refinement of $C$, and that $x_0$ is nonsingular with respect to both $C$ and $C'$. By the cocycle property, each term in $S(C)(x_0)$ is a finite sum of terms from $S(C')(x_0)$. Since both sums converge absolutely, this implies they have the same sum. The singular case is handled by considering sums over stars as above, and we leave the details to the reader.

Now if $C'$ is any good fan, we can construct an $F$-rational common refinement $C''$ of $C$ and $C'$. This will also be a good fan, although perhaps with respect to a different subgroup

$V''$. The previous argument shows $S(C)(x_0) = S(C'')(x_0) = S(C')(x_0)$, which completes the proof. ∎

We are now ready to state our main result:

**Theorem 1.** *Let $C$ be a good fan (§2.3), and suppose that $x_0 \in F$ is totally positive. Then*

$$S(C)(x_0) = \frac{1}{\mathrm{N}(x_0)}.$$

The proof will occupy the rest of the article.

## 3. Chains and cycles in linear varieties

**3.1.** Let $X$ be a linear variety of dimension $n$, that is $X$ is an algebraic variety biregular with $\mathbf{P}^n$. Since the automorphism group of $\mathbf{P}^n$ is $PGL(n+1)$, any concept that involves the linear structure of $\mathbf{P}^n$ is meaningful for $X$. In particular, if we fix an isomorphism $\phi : \mathbf{P}^n \to X$, we may define a line in $X$ to be the image under the map $\phi$ of a line in $\mathbf{P}^n$. The definition does not depend on the choice of $\phi$ and thus the concept of a line (plane, hyperplane, etc.) is intrinsic to $X$.

If $X$ is a linear variety, then we define $X^*$ to be the set of hyperplanes in $X$. Then $X^*$ has a canonical structure of linear variety.

**3.2.** We wish to define $C_k(X)$ and $Z_k(X)$, the chains and cycles of dimension $k$.

Let $C_0(X) = \mathbf{Z}[X]$, that is, $C_0(X)$ is the free abelian group generated by the points of $X$. Thus $c_0 \in C_0$ is a map $c_0\colon X \to \mathbf{Z}$ with the property $c_0(x) = 0$ for all but finitely many $x$. Let $\partial_0\colon C_0(X) \to \mathbf{Z}$ be the map $\partial_0(c_0(x)) = \sum_{x\in X} c_0(x)$, and let $Z_0 = \ker \partial_0$.

We define $C_k(X)$ inductively:

$$C_k(X) \;=\; \bigoplus_{L^{(k)}\subseteq X} Z_{k-1}(L^{(k)}),$$

where $L^{(k)}$ ranges over all linear subvarieties of dimension $k$, and $Z_{k-1}(L^{(k)})$ is the group of $k-1$ cycles in $L^{(k)}$. A typical element of $C_k(X)$ has the form $c_k = c_k(X) = (z_{k-1}(L^{(k)}))$, where $z_{k-1}(L^{(k)})$ is a $k-1$ cycle in $L^{(k)}$ that is zero for all but finitely many $L^{(k)}$.

If $Y, X$ are linear varieties with $Y \subseteq X$, then we have a canonical map $Z_{k-1}(Y) \hookrightarrow Z_{k-1}(X)$, which we denote $z_{k-1}(Y) \mapsto z_{k-1}(Y, X)$. Using this we define the differential $\partial_k\colon C_k(X) \to C_{k-1}(X)$ by

$$\partial_k(c_k) \;=\; \partial_k((z_{k-1}(L^{(k)})) \;=\; \sum_{L^{(k)}\subseteq X} z_{k-1}(L^{(k)}, X),$$

and put $Z_k(X) = \ker(\partial_k)$. Note that $\partial_k \circ \partial_{k-1} = 0$.

**3.3.** We now give some examples of our construction.

(1) The cycle group $Z_0(L)$ is generated by elements of the form $w - v$ where $v, w \in L$ (here $L \subseteq X$ is any linear subvariety). The only relations are $(w-v)+(v-u) = (w-u)$ and $(w-v) = -(v-w)$, where $u, v, w \in L$.

(2) The chain group $C_1(X)$ is generated by symbols of the form $\langle w - v\rangle$, with $w, v \in X$. The only relations are $\langle w-v\rangle = -\langle v-w\rangle$ and $\langle w-v\rangle + \langle v-u\rangle = \langle w-u\rangle$ where $w, v, u \in X$ are collinear. If $c_1 = \sum a_i \langle w_i - v_i\rangle \in C_1(X)$, note that $\partial_1(c_1) = \sum a_i(w_i - v_i) \in C_0(X)$.

(3) Every element of $Z_1(X)$ can be written in the form $\sum_{i=1}^{r} \langle v_{i+1} - v_i\rangle$, where $v_{r+1} = v_1$.

**3.4.** Now we discuss simplicial cycles. Let $A_0, \ldots, A_k \in X$ be linearly independent (the concept of linear independence is intrinsic to $X$). Then we define $\sigma_{k-1}(A_0, \ldots, A_k)(X) \in Z_{k-1}(X)$ as follows:

(1) If $k = 1$, then $\sigma_0(A_0, A_1) := A_0 - A_1$.

(2) If $k > 1$, then

$$\sigma_{k-1}(A)(X) = \sigma_{k-1}(A_0, \ldots, A_k)(X) := \sum_{r=0}^{k} (-1)^r \sigma_{k-2}(A_0, \ldots, \hat{A}_r, \ldots A_k)(L_r),$$

where $L_r$ is the linear space spanned by $A_0, \ldots, \hat{A}_r, \ldots, A_k$.

(3) If $A_0, \ldots, A_k \in X$ are linearly dependent. then $\sigma_{k-1}(A_0, \ldots, A_k)(X) := 0$.

**Proposition 3.**

(1) $\sigma_{k-1}(A)(X) \in Z_{k-1}(X)$.

(2) *For every permutation $\pi$ on $k+1$ letters, we have*

$$\sigma_{k-1}(A_{\pi(0)}, \ldots, A_{\pi(k)}) \;=\; \mathrm{sign}(\pi)\sigma_{k-1}(A_0, \ldots, A_k).$$

(3) *$Z_{k-1}(X)$ is spanned by simplices.*

*Proof.* The proofs of (1) and (2) are standard. For (3), let $z_{k-1}(X) = (z_{k-2}(L)) \in Z_{k-1}(X)$. Then by induction

$$z_{k-2}(L) \;=\; \sum_{A \in \mathbf{A}(L)} \sigma_{k-2}(A)(L),$$

where $\mathbf{A}(L)$ is a finite set of $k$ tuples in $L$. Now let $A_0 \in X$ be arbitrary. Then we can check that

$$z_{k-1}(X) \;=\; \sum_{L} \sum_{A \in \mathbf{A}(L)} \sigma(A_0, A)(X).$$

■

## 4. CPD functions

**4.1.** Let $G$ be a torsion free abelian group and let $f\colon X^{k+1} \to G$ be a function. Then we say that $f$ is a *CPD function* if it satisfies the *cocycle* property, the *permutation* property, and the *degeneracy* property:

**C.** For every $A_0, \dots, A_{k+1} \in X$, we have

$$\sum_{r=0}^{k+1} (-1)^r f(A_0, \dots, \hat{A}_r, \dots, A_{k+1}) = 0.$$

**P.** For every permutation $\pi$ on $k+1$ letters, we have

$$f(A_{\pi(0)}, \dots, A_{\pi(k)}) = \operatorname{sign}(\pi) f(A_0, \dots, A_k).$$

**D.** If $(A_0, \dots, A_k)$ are linearly dependent, then $f(A_0, \dots, A_k) = 0$.

**4.2.** Here are some examples of CPD functions.

(1) Let $X = \mathbf{P}^{n-1}$ and let $G = \mathbf{R}(x)$, the field of rational functions in $n$ real variables, viewed as an abelian group with respect to addition. The function $h\colon (\mathbf{R}^n)^n \to \mathbf{R}(x)$ from §2.4 is homogeneous in the points $(A_1, \dots, A_n)$. Hence it induces a function $h\colon (\mathbf{P}^{n-1})^n \to \mathbf{R}(x)$, and this is a CPD function.

(2) Let $X$ be a linear variety of dimension $n$, let $X^*$ be the dual variety, and let $G = Z_{n-1}(X^*)$. We define a CPD function $D\colon X^{n+1} \to G$ as follows. If $A_0, \dots, A_n$ are linearly dependent, then $D(A_0, \dots, A_n) := 0$. Otherwise we let $D(A_0, \dots, A_n) := \sigma_{n-1}(B_0, \dots, B_n)$, where $B_i \in X^*$ is the linear space spanned by $A_0, \dots, \hat{A}_i, \dots, A_n$.

The following proposition shows that a $G$-valued CPD function $f$ extends to give a $G$-valued cocycle $\tilde{f}$ on the cycle group $Z_{k-1}(X)$. This will play a key role in the proof of Theorem 1.

**Proposition 4.** *Let $f\colon X^{k+1} \to G$ be a CPD function. Then there exists a unique homomorphism $\tilde{f}\colon Z_{k-1}(X) \to G$ satisfying $\tilde{f}(\sigma(A_0, \dots, A_k)(X)) = f(A_0, \dots, A_k)$ for all $A_0, \dots, A_k \in X$.*

*Proof.* Let $\tilde{f}$ be the function defined on simplices in $Z_{k-1}(X)$ as in the statement of the proposition. Then $\tilde{f}$ can be defined on any cycle $z \in Z_{k-1}(X)$ by writing $z = \sum_A \sigma(A)$, where the $\sigma(A)$ are simplices, and then putting $\tilde{f}(z) = \sum_A \tilde{f}(\sigma(A))$. That this extension is well-defined follows immediately from the cocycle property of the CPD function $f$. ■

**Definition 1.** Suppose $D$ is the CPD function in the second example above. Then the extension $\tilde{D}$ gives a map $Z_{k-1}(X) \to Z_{k-1}(X^*)$. We shall write $\tilde{D}((z_{k-1})(X)) = [z_{k-1}(X)]^*$, and shall say that $[z_{k-1}(X)]^*$ is the *dual cycle* to $z_{k-1}(X)$.

## 5. Convex Polyhedra

**5.1.** We now extend some of the basic notions of convex geometry in affine space to the linear variety $X$. An *open half space* of $X$ is a connected component of the set $X \setminus (H_1 \cup H_2)$, where $H_1, H_2 \subseteq X$ are two different hyperplanes. A *closed half space* is the closure of an open half space. A *convex polyhedron* $K \subset X$ is a finite intersection of closed half spaces with the property $K \cap H = \emptyset$ for some hyperplane $H$. Thus a convex polyhedron is just the image of a usual compact convex polyhedron in $\mathbf{R}^{n+1} \setminus \{0\}$ via the map $\mathbf{R}^{n+1} \setminus \{0\} \to \mathbf{P}^n \to X$. Moreover, there is similarly an obvious bijection between polyhedral cones in $\mathbf{R}^{n+1}$ and convex polyhedra in $X$.

Let $H \subseteq X$ be a hyperplane. If $v_1, \ldots, v_r$ are points in the affine space $\in X \setminus H$, then the convex hull of $\{v_1, \ldots, v_r\}$ in $X \setminus H$ (in the usual sense of convex geometry) is a convex polyhedron in $X$. We can also define the convex hull of a set of points in $X$ without passing to an affine subset, simply by applying the usual definition with our notion of half space. If $r = k + 1$ and if the $v_i$ are linearly independent, then we call the resulting convex hull a *simplicial polyhedron*.

**5.2.** Now let $K \subseteq X$ be a convex polyhedron with $\dim(K) = \dim(X) = k$. Let $K^* \subseteq X^*$ be the set of hyperplanes in $X$ that do not intersect the interior of $K$. Then $K^*$ is called the *dual* of $K$, and is itself a convex polyhedron in $X^*$. It is easy to check that if $K$ is a convex polyhedron corresponding to the polyhedral cone $\sigma_K \in \mathbf{R}^{n+1}$, then the dual convex polyhedron $K^*$ corresponds to the dual cone $\sigma_K^*$.

An equivalent definition of $K^*$ is as follows. Let $h$ by any fixed point in the interior of $K$. Then $h^*$, the set of hyperplanes of $X$ that pass through $h$, is a hyperplane in $X^*$. Let $\Phi_1, \ldots, \Phi_t$ be the faces of $K$. Then the $\Phi_i$ span hyperplanes $H_i$ that do not contain $h$ and thus determine points $H_i^* \in X^* \setminus h^*$. Then $K^*$ is the convex hull of $F_i^*$ inside $X^* \setminus h^*$.

The faces of $K^*$ are convex polyhedra in 1–1 correspondence with the non-degenerate vertices of $K$ (these are the vertices not contained in any of the planes spanned by the proper faces of $K$). If $v$ is such a vertex, then the face $F_v^*$ corresponding to $v$ is the convex polyhedron in $v^*$ whose vertices are the hyperplanes spanned by the faces of $K$ containing $v$. Thus $F_v^*$ is the convex hull of those vertices, taken inside $v^* \setminus h^*$.

**5.3.** There is another procedure for constructing $F_v^* \subseteq v^*$ that is useful for inductive proofs. Let $v \in K$ be a non-degenerate vertex. Then there exists a hyperplane $H_v \subseteq X$, called a *vertex hyperplane*, with the following properties:

(1) $v \notin H_v$.

(2) If $F_v = K \cap H_v$, then $K \setminus H_v$ consists of two connected components. One of those components has, as its closure, the convex polyhedron given by the convex hull of $F_v$, and the point $v$.

The convex polyhedron $F_v^*$ is the dual of $F_v$ in the sense of §5.2. To be precise, we have a canonical map $H_v^* \to v^*$ that associates to a hyperplane $L \subset H_v$ the hyperplane of $X$ spanned by $v$ and $L$. Then the dual of $F_v$, which is a convex polyhedron in $H_v^*$, is identified with $F_v^*$ via the canonical map $H_v^* \to v^*$.

## 6. The cycle associated to a polyhedron

**6.1.** Let $K \subseteq X$ be an oriented convex polyhedron of dimension $k$. Then we associate to the boundary of $K$ a cycle $z_{k-1}(K)(X) \in Z_{k-1}(X)$ as follows. If $K$ is 1 dimensional, then $\partial(K)$ is just a pair of points $x, y \in X$. The orientation allows us to assign one of the points, say $x$, the value $+1$, and the other point $y$ the value $-1$. Then we define $z_0(K)$ to be $x - y$.

In general, we define $z_{k-1}(K)$ inductively as follows. For each face $F \subseteq K$, let $L_F$ be the linear space spanned by $F$. Then $F$ is a convex polyhedron of dimension $k-1$ embedded in $L_F$. Thus $z_{k-2}(F)(L_F)$ has been defined as a $k-2$ cycle in $L_F$. We now define

$$z_{k-1}(K) = \sum_F z_{k-2}(F)(L_F).$$

In particular,

$$z_{k-1}(K^*) \;=\; \sum_v z_{k-2}(F_v^*) \;=\; \sum_v z_{k-2}([K \cap H_v])^*). \tag{1}$$

A cycle of the form $z_{k-1}(K)$ is called a *polyhedral cycle*.

At this stage we have two notions of duality for a polyhedral cycle $z(K)$: the cycle $z(K^*)$ associated to the dual convex polyhedron $K^*$, and the dual cycle $[z(K)]^*$ constructed using the CPD function $\tilde{D}$ from Example 2 of §4.2. The following theorem asserts that these two notions coincide.

**Theorem 2.** *Let $K \subseteq X$ be a convex polyhedron. Then we have*

$$z_{k-1}(K^*) \;=\; [z_{k-1}(K)]^*.$$

*Proof.* Without loss of generality, we may assume the faces of $K$ are simplicial polyhedra. Choose a point $u$ in the interior of $K$. Then $K = \cup_F c(u, F)$ is a decomposition into simplicial polyhedra, where $c(u, F)$ is the cone on $F$ with vertex $u$, and $F$ ranges over all faces of $K$. We obtain

$$z_{k-1}(K) = \sum_F z_{k-1}(c(u, F)),$$

and thus

$$[z_{k-1}(K)]^* = \sum_F [z_{k-1}(c(u, F))]^*.$$

On the other hand, we have

$$z_{k-1}(K^*) \;=\; \sum_v z_{k-2}([K\cap H_v])^*)$$

Hence we must show

$$\sum_v z_{k-2}([K\cap H_v])^*) \;=\; \sum_F [z_{k-1}(c(u,F))]^* \;=\; \sum_F z_{k-1}(c(u,F)^*), \tag{2}$$

where $H_v$ is a vertex hyperplane. Here the second equality follows since $c(u,F)$ is a simplicial polyhedron, and our theorem is trivially true for such polyhedra.

Each side of (2) is a cycle in $X^*$, and thus each side has a $v^*$ component for all $v \in X$. We shall compare the $v^*$ components for each $v$ and show they are equal. Fix $v \in X$. We may assume $v$ is a non-degenerate vertex of $K$, for otherwise the $v^*$ component of each side of (2) vanishes. Applying (1) with $K$ replaced by $c(u,F)$, and substituting the result in (2), we are reduced to showing

$$z_{k-2}([K\cap H_v]^*) \;=\; \sum_{\{F|v\in F\}} z_{k-2}([c(u,F)\cap H_v]^*).$$

But we clearly have

$$z_{k-2}([K\cap H_v]) \;=\; \sum_{\{F|v\in F\}} z_{k-2}([c(u,F)\cap H_v]), \tag{3}$$

since $K\cap H_v = \cup_{\{F|v\in F\}}[c(u,F)\cap H_v]$ is a decomposition into simplicial polyhedra. Taking duals of both sides of (3), and using induction on $k$, completes the proof. ∎

## 7. Cycles and the function $h^*$

**7.1.** Now we want to apply the machinery in §§3–6 to compute the sum $S(C)(x_0)$ for totally positive $x_0 \in F$. Let $X$ be the projective space $\mathbf{P}^{n-1}$. Recall that we have identified $\mathbf{R}^n$ with its dual. This allows us to identify $X$ with its dual $X^*$, and thus to identify $Z_k(X)$ with $Z_k(X^*)$. We will do this throughout the following discussion. Also, from this point on we will only need to consider the cycle group $Z_{n-2}(X)$, and so to lighten notation we will drop the degree subscripts if no confusion is possible.

**7.2.** Let $t \in C(n)$ be a top dimensional cone. The results from the previous sections show that $t$ determines a polyhedron $K_t$ and a cycle $z(K_t) \in Z(X)$. If $t^*$ is the dual cone with associated polyhedron $K_t^*$, then we have

$$(z(K_t))^* = z(K_t^*),$$

where the star on the left denotes the dual cycle construction. Let $\tilde{h}\colon Z(X) \to \mathbf{C}(x)$ be the extension to $Z(X)$ of the function described in Example (1) from §4.2 (whose existence is guaranteed by Proposition 4). Then we have

$$h^*(t)(x) = \tilde{h}(z(K_t)^*).$$

In particular, let $T \subset C(n)$ be any finite subset, and consider the sum

$$S(T)(x) := \sum_{t \in T} h^*(t)(x).$$

Let $z(T)$ be the cycle $\sum_{t \in T} z(t)$. Then it follows that

$$S(T)(x) = \tilde{h}(z(T)^*)(x).$$

**7.3.** Now suppose

$$\Sigma := \bigcup_{t \in T} t$$

is *itself* a convex polyhedral cone. In general, $\Sigma$ will not be simplicial, but there is nevertheless a well-defined cycle $z(\Sigma) \in Z(X)$ associated to $\Sigma$, namely that which is induced by the polyhedron $K_\Sigma$. The next lemma follows easily from the previous discussion and the formal properties of our cycle apparatus. We omit the simple proof.

**Lemma 2.** *We have*

$$S(T)(x) = \tilde{h}(z(K_\Sigma)^*)(x).$$

## 8. Admissible units

**8.1.** To this point we have not used the fact that $C$ admits an action by a finite-index subgroup $V$ of $U$, the group of totally positive units. In this and the next section we use this structure to construct a new fan $C'$ with $S(C)(x) = S(C')(x)$, and then to construct a sequence of partial sums for $S(C')(x)$. Then in the final section we apply the cycle machinery and Hurwitz's geometric interpretation of $h$ to complete the proof of Theorem 1. We begin by discussing some special collections of units in $V$. From now on, we assume that $n \geq 3$, since in the quadratic case a direct proof of the main theorem is easy.

**8.2.** Recall that $E = (e_1, \ldots, e_n)$ is the canonical basis of $\mathbf{R}^n$. Let $\varepsilon \in U$ be a totally positive unit. The *limit pair* $L(\varepsilon)$ is the pair of projective points $(\varepsilon(-\infty), \varepsilon(\infty)) \subset \mathbf{P}^{n-1} \times \mathbf{P}^{n-1}$, where

$$\varepsilon(\alpha) := \lim_{t \to \alpha} \varepsilon^t.$$

It is easy to see that $\varepsilon(\alpha)$ always has the form $\sum_{i\in I} e_i$, where we abuse notation slightly and use the same symbol for a point in $\mathbf{R}^n$ and the point it induces in $\mathbf{P}^{n-1}$. We say $\varepsilon$ is *generic* if its limit pair has the form $(e_i, e_j)$, where $i \neq j$.

Let $[\![n]\!] = \{1, \ldots, n\}$, and let $\mathcal{I}$ be the set of all subsets of $[\![n]\!]$ of order $(n-1)$. Recall that for any $i \in [\![n]\!]$ and $x \in F$, we denote $v_i(x)$ by $x^{(i)}$. In what follows indexing subscripts and superscripts referring to the real places of $F$ will be taken modulo $n$.

**Definition 2.** Let $T = \{\varepsilon_1, \ldots, \varepsilon_n\} \subseteq U$ be a set of totally positive units such that for any $I \in \mathcal{I}$, the subset $T_I = \{\varepsilon_i \mid i \in I\}$ is independent (i.e., the regulator of $T_I$ is nonzero). We say that $T$ is *admissible* if the following hold:

(1) For each $\varepsilon \in T$, the coordinates $(\varepsilon^{(1)}, \ldots, \varepsilon^{(n)})$ are distinct.

(2) We have $L(\varepsilon_i) = (e_i, e_{i+1})$.

(3) We have $L(\varepsilon_i/\varepsilon_j) = (e_i, e_j)$ for $i \neq j$.

**Lemma 3.** *Let $\varepsilon_1, \ldots, \varepsilon_n \in V$ and let $T = \{\varepsilon_1, \ldots, \varepsilon_n\}$. Suppose that $T$ satisfies the following: there exist real numbers $b > a > 1$ such that for each $\varepsilon_i \in T$, we have*

(1) $\varepsilon_i^{(i)} < 1$, $\varepsilon_i^{(j)} > 1$ *if* $j \neq i$;

(2) $\varepsilon_i^{(i+1)} > \varepsilon_i^{(i+2)} > \cdots > \varepsilon_i^{(i-1)}$;

(3) $\varepsilon_i^{(j)}/\varepsilon_i^{(k)} \in (a^{-1}, a)$ *for all* $j, k \neq i$; *and*

(4) $\varepsilon_i^{(j)}/\varepsilon_i^{(i)} > b$, *for all* $j \neq i$.

*Then $T$ is admissible.*

*Proof.* The first admissibility condition is clearly satisfied, and we need only check that limit pairs behave as desired. The condition $L(\varepsilon_i) = (e_i, e_{i+1})$ is obvious. The condition $L(\varepsilon_i/\varepsilon_j) = (e_i, e_j)$ follows since in the ratio $\mu = \varepsilon_i/\varepsilon_j$, the smallest (resp. largest) component is $\mu^{(i)}$ (resp. $\mu^{(j)}$). ∎

**Proposition 5.** *Any finite-index $V \subset U$ contains an admissible set of units. In fact, for every $b > a > 1$, there exists $T \subseteq V$ satisfying the hypothesis of lemma 3.*

*Proof.* Consider the standard map $\log\colon \Delta_\infty \to \mathbf{R}^n$ given by $x \mapsto (\log x^{(1)}, \ldots, \log x^{(n)})$. The group $V$ is taken to a discrete subgroup $L(V) \subset \mathbf{R}^n$, which we may view as a lattice in the hyperplane $H = L(V) \otimes \mathbf{R}$. This also endows $H$ with a $\mathbf{Q}$-structure, namely $H(\mathbf{Q}) = L(V) \otimes \mathbf{Q}$.

Now fix $b > a > 1$ and consider what conditions the hypotheses of Lemma 3 become in the subspace $H$. Let $(\xi_1, \ldots, \xi_n)$ be the coordinates of a point in $\mathbf{R}^n$ with respect to $E$, so that $H$ is defined by the equation $\sum \xi_i = 0$. (Note that the rational structure induced by $E$ is *not* the same as that induced by $L(V)$.) We find that a set of units $\{\varepsilon_1, \ldots, \varepsilon_n\}$

satisfies (1)–(4) of the statement if and only if for each $i$, the point $(\log \varepsilon_i^{(1)}, \ldots, \log \varepsilon_i^{(n)})$ lies in the region $R_i \subset H$ determined by the inequalities

(1) $\xi_i < 0$, $\xi_j > 0$ for $j \neq i$;

(2) $\xi_{i+1} > \xi_{i+2} > \cdots > \xi_{i-1}$;

(3) $-\log a < \xi_j - \xi_k < \log a$, for all $j, k \neq i$; and

(4) $\xi_j - \xi_i > \log b$, for all $j \neq i$.

We claim that each $R_i$ is an unbounded open set in $H$ of full dimension. This can be seen as follows. For fixed $i$, the conditions (1) define an $(n-1)$-dimensional simplicial cone

$$\sigma_i = \Bigl\{\sum_{i \neq j} \lambda_j (e_j - e_i) \Bigm| \lambda_j \in \mathbf{R}_{\geq 0}\Bigr\}.$$

The conditions (2) cut out an $(n-1)$-cone $\sigma_i'$ in the barycentric subdivision of $\sigma_i$. Let $\tau_i$ be the barycenter of $\sigma_i$, i.e. the 1-cone

$$\tau_i = \mathbf{R}_{\geq 0}(-(n-1)e_i + \sum_{j \neq i} e_j),$$

and let $U_i$ be a small tubular neighborhood of $\tau_i$. Then each inequality in (3) determines a half-space containing $U_i$. Hence the intersection of these half-spaces with $\sigma_i'$ is unbounded. Finally, the half-spaces determined by (4) include all points in $U_i$ that are sufficiently far away from the origin, and hence the region $R_i$ is unbounded and has dimension $n-1$.

To conclude the proof, we claim that each $R_i$ contains infinitely many points of $L(V)$. Indeed, we assert that for each $i$, we have $L(V) \cap \tau_i = \{0\}$. This proves the claim, since it implies the image of $L(V)$ is dense in the quotient $H/\mathbf{R}\tau_i$, and hence the inverse image of any open set in this quotient must contain infinitely many points of $L(V)$.

To prove the assertion, let $x \in L(V) \cap \tau_i$ be nonzero. Then we may write

$$x = (\lambda, \ldots, \lambda, -(n-1)\lambda, \lambda, \ldots, \lambda),$$

for some positive real number $\lambda$, where the $-(n-1)\lambda$ appears in the $i$th position. This implies that there is a unit $\varepsilon$ of the form

$$\varepsilon = (e^\lambda, \ldots, e^\lambda, e^{-(n-1)\lambda}, e^\lambda, \ldots, e^\lambda);$$

in other words, all $v_j(\varepsilon)$ with $j \neq i$ are equal. We claim that no element of $F$ that is not in $\mathbf{Q}$ can have this form. To see this, let $\alpha \neq \beta$ be the distinct infinite places of $\varepsilon$, and let $f(X) = (X-\alpha)^{n-1}(X-\beta) \in \mathbf{R}[X]$. Clearly $f(\varepsilon) = 0$, and so the minimal polynomial $g(X)$ of $\varepsilon$ must divide $f$. This implies $g(X) = (X-\alpha)(X-\beta)$, which implies $\varepsilon \in F$ is quadratic over $\mathbf{Q}$. But this means that under $F \to \mathbf{R}^n$, half of the embeddings must equal $\alpha$ and half must equal $\beta$. This contradicts the assumption that $n \geq 3$ and the proof of the proposition is complete. ∎

## 9. Partial sums

**9.1.** Let $V$ be a fixed subgroup of finite index in $U$, the group of totally positive units. Choose and fix an admissible set of units $T = \{\varepsilon_1, \ldots, \varepsilon_n\}$ satisfying the conditions of Lemma 3, with $a, b$ chosen so that $b > a^n > 1$. For each $I \in \mathcal{I}$, define $V_I$ by

$$V_I = \{\prod_{i\in I} \varepsilon_i^{\alpha_i} \mid \alpha_i \in \mathbf{Z}, \sum_{i\in I} \alpha_i = 0\}.$$

Let $\Sigma_I \subset \Delta_\infty$ be the convex cone generated by the half-lines through the points $V_I$.

**Lemma 4.** *The points in $V_I$ generate the spanning rays of $\Sigma_I$. In other words, no point of $V_I$ lies in the relative interior of $\Sigma_I$.*

*Proof.* Let $j = [\![n]\!] \setminus I$ and let $\varphi_j \colon \Delta_\infty \to \mathbf{R}_+^{n-1}$ be the map

$$\varphi_j(x_1, \ldots, x_n) \;=\; \left(\frac{x_1}{x_j}, \ldots, \frac{\widehat{x_j}}{x_j}, \ldots, \frac{x_n}{x_j}\right)$$

where the $j^{th}$ component is omitted. Since $\varphi_j$ takes straight lines to straight lines, we see that the image $\Pi_I = \varphi_j(\Sigma_I)$ is a convex polyhedral subset of $\mathbf{R}_+^{n-1}$ with $\varphi_j^{-1}(\Pi_j) = \Sigma_I$. To prove the lemma it thus suffices to show that the points

$$\{\varphi_j(v) \bigm| v \in V_I\} \tag{4}$$

are the vertices of $\Pi = \Pi_I$.

Now consider the set $\partial\Pi_{\mathbf{R}}$ of the boundary of the "real points" of $\Pi$, by which we mean the subset in $\mathbf{R}^{n-1}$ of points of the form

$$\partial\Pi_{\mathbf{R}} := \{\varphi_j\left(\prod_{i\in I} \varepsilon_i^{t_i}\right) \mid t_i \in \mathbf{R},\; \sum_{i\in I} t_i = 0\}.$$

We claim that

$$\partial\Pi_{\mathbf{R}} = \{(z_1, \ldots, z_{n-1}) \bigm| z_i > 0, \prod z_i^{a_i} = 1\},$$

where $(a_1, \ldots, a_{n-1})$ are certain fixed real numbers. To see this, assume (to ease notation) that $j = n$, and let $Z$ (respectively $T$) be the column vector ${}^t(z_1, \ldots, z_{n-1})$ (resp. ${}^t(t_1, \ldots, t_{n-1})$). Then $Z \in \partial\Pi_{\mathbf{R}}$ if and only if $\log Z = E \cdot T$ for some $T$, where $E$ is the matrix whose $p, q$ entry is

$$(\log \varepsilon_q^{(p)} - \log \varepsilon_q^{(j)}), \quad p, q = 1, \ldots, n-1.$$

Since the $\varepsilon_i$ generate a subgroup of the unit group with full rank, the matrix $E$ is invertible. This proves the existence of the $a_i$. In fact, we have $(a_1, \ldots, a_{n-1}) = (1, \ldots, 1)E^{-1}$. The same argument works for any $j = 1, \ldots, n-1$, which proves the claim.

Next we claim that all the $a_i$ are positive: If not, then we can rewrite $\prod z_i^{a_i} = 1$ as $\prod_{k\in A} z_k^{r_k} = \prod_{l\in B} z_l^{s_l}$ where $I = A \cup B$ is a partition into disjoint non-empty subsets, and the $r_k, s_l$ are all positive real numbers. Let $r_{k_0} = \max r_k$ and $s_{l_0} = \max s_l$, and let $z = \varphi(\varepsilon_{l_0}/\varepsilon_{k_0})$. Then $z \in \partial\Pi_{\mathbf{R}}$. Moreover, $z_{k_0} > (ba^{-1})^{r_{k_0}}$ and $z_k > a^{-r_{k_0}}$ if $k \neq k_0$. Since $b > a^n$, this shows $\prod_{k\in A} z_k^{r_k} > 1$. Similarly, $\prod_{l\in B} z_l^{s_l} < 1$. This is a contradiction, and thus we conclude that all the $a_i$ are positive.

Let $\Pi_{\mathbf{R}}$ be defined as

$$\Pi_{\mathbf{R}} = \bigl\{(z_1, \dots, z_{n-1}) \bigm| z_i > 0, \prod z_i^{a_i} \geq 1\bigr\},$$

We claim that $\Pi_{\mathbf{R}}$ is a convex subset of $\mathbf{R}^{n-1}$. Note first that $\partial\Pi_{\mathbf{R}}$ is the graph of the function $f\colon \mathbf{R}^{n-2} \to \mathbf{R}$ defined by

$$f(z_1, \dots, z_{n-2}) = \prod_{i=1}^{n-2} z_i^{-p_i}, \quad p_i > 0.$$

Now consider the Hessian matrix

$$M = \Bigl(\frac{\partial^2 f}{\partial z_i \partial z_j}\Bigr).$$

We claim that $M$ is positive-definite, from which the convexity of $\Pi_{\mathbf{R}}$ follows immediately. To see this, for $k = 1, \dots, n-2$ let $M_k$ be the $k \times k$ upper-left submatrix of $M$. Then by an induction argument

$$\det M_k = (1 + \sum_{i\leq k} p_i) \prod_{i\leq k} p_i z_i^{-(n-2)p_i - 2}. \tag{5}$$

Since $p_i, z_i > 0$, this determinant is positive on $\partial\Pi_{\mathbf{R}}$ for each $k$. By a standard result of linear algebra, this implies $M$ is positive-definite, and thus $\Pi_{\mathbf{R}}$ is a convex subset of $\mathbf{R}^{n-1}$.

We claim convexity of $\Pi_{\mathbf{R}}$ implies that the points in (4) are exactly the vertices of $\Pi$. To see this, assume that $\varphi_j(v)$ is in the relative interior of $\Pi$ for some $v \in V_I$. Since $\Pi_{\mathbf{R}}$ is convex, we deduce that $\Pi \subseteq \Pi_{\mathbf{R}}$ and hence $\varphi_j(v)$ is in the relative interior of $\Pi_{\mathbf{R}}$. But $\varphi_j(v) \in \partial\Pi_{\mathbf{R}}$. This contradiction completes the proof. ∎

**Lemma 5.**

(1) *The faces on the* $n-1$ *dimensional convex polyhedron* $\Pi_I \subseteq \mathbf{R}_+^{n-1}$ *are compact polyhedra of dimension* $n-2$.

(2) *If we let* $\mathcal{F}_I$ *denote the set of faces of* $\Pi_I$*, then there exists a finite subset* $S_I \subseteq \mathcal{F}_I$ *with the following property: For every* $f \in \mathcal{F}_I$ *there exists* $\varepsilon \in V_I$ *such that* $f = \varepsilon \cdot s$ *for some* $s \in S_I$.

*Proof.* Let $f \in \mathcal{F}$, let $H$ be the hyperplane spanned by $f$, and $v \in f \subseteq H$ be a vertex of $f$. For each $i = 1, \dots, n$, choose a $\varepsilon_i \in V_I$ that is very close to the standard basis element

$e_i$ inside $\mathbf{P}^{n-1}$ (in other words, the $i^{th}$ component of $\varepsilon_i$ is much larger than all the other components). For $r$ a positive integer, we let $K(r)$ be the hyperplane spanned by the $\varepsilon_i^r$. We see that $K(r) \cap \mathbf{R}_{\geq 0}^{n-1}$ is a compact simplex of dimension $n-2$.

Let $\tilde{K}(r)$ be the $n-1$ simplex spanned by $K(r)$ and the origin. Then $\tilde{K}(r)$ is compact and contains $H \cap \mathbf{R}_{\geq 0}^{n-1}$ for $r$ sufficiently large. Thus shows that $H \cap \mathbf{R}_{\geq 0}^{n-1}$ is a compact simplex, and thus $H \cap \Pi$ is compact, which proves the first part of Lemma 5.

Now we note that $V_I$ acts on $\partial \Pi_{\mathbf{R}}$ and that $\partial \Pi_{\mathbf{R}} / V_I$ is compact. Thus we can choose $r$ such that $\tilde{K}(r)$ contains a compact fundamental domain $D \subseteq \partial \Pi_{\mathbf{R}}$. Now let $f \in \mathcal{F}_I$, let $v \in f$ be a vertex, and choose $\varepsilon \in V_I$ such that $\varepsilon \cdot v \in D$. Then $\varepsilon \cdot f$ is a face of $\partial \Pi$, and by the proof of part one, $\varepsilon \cdot f \subseteq \tilde{K}(r')$ for some $r' > r$ that depends on $D$ but not on $f$. Thus all the vertices of $\varepsilon \cdot f$ lie in $\tilde{K}(r') \cap \Pi_R$, which is compact. Since $V_I \cap \tilde{K}(r') \cap \Pi_R$ is finite, the second part of the lemma is proved. ∎

**9.2.** Now we come to the main tool that we need to prove Theorem 1.

**Theorem 3.** *Let $C$ be a good fan, and $V \subseteq U$ a subgroup of finite index acting on $C$. Then there exists $C'$, a good refinement of $C$, and a family of convex subsets $\Sigma_N \subseteq \Delta_\infty$, with the following properties:*

1. *For each $N$, the set $\Sigma_N$ is a finite union of top dimensional simplices in $C'$.*
2. *$\Delta_\infty = \bigcup_N \Sigma_N$.*

*Proof.* Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ satisfy the conditions of Lemma 3, with $a, b$ chosen so that $b > a^n > 1$. For each $j, 1 \leq j \leq n$, let $I(j) = \{i \mid 1 \leq i \leq n, i \neq j\}$. Define

$$\Sigma_N = \bigcap_{J=1}^{N} \varepsilon_j^{-N} \cdot \varphi_j^{-1}(\Pi_{I(j)})$$

Now fix $j$, and let $I = I(j)$. Let $\mathcal{D}_j = \varphi_j(C)$, which is a simplicial decomposition of $\mathbf{R}^{n-1}$. Choose a fundamental domain $F$ for the action of $V$ on $\mathbf{R}^{n-1}$ of the form

$$F = \bigcup_{t \in T} t,$$

where $T$ is a finite subset of top dimensional simplices in $\mathcal{D}$, and let $S_I$ be as in Lemma 5. Since $S_I$ is finite, and since, by Lemma 5, the elements of $S_I$ are compact, the set

$$\big\{(\varepsilon, t, s) \in V \times T \times S_I \bigm| \varepsilon \cdot s \cap t \neq \emptyset\big\}$$

is finite. Thus there is a simplicial decomposition $\mathcal{D}'_j$, which is a refinement of $\mathcal{D}_j$, with the property that every $f \in \mathcal{F}_I$ is a finite union of $n-2$ simplices in $\mathcal{D}'_j$.

Let $C'$ be a common refinement of $\varphi_J^{-1}(\mathcal{D}'_j)$ for all $j$. Then $C'$ clearly satisfies condition 1 of Theorem 3, and $\Sigma_N$ clearly satisfies condition 2. ∎

## 10. Proof of the main theorem

We now complete the proof of Theorem 1. Let $x_0 \in F$ be totally positive, and consider the sequence of cones $\{\Sigma_N\}$ constructed in Theorem 3. By Theorem 3 we have $\lim_{N\to\infty} S(\Sigma_N)(x_0) = S(C')(x_0)$, which in turn equals $S(C)(x_0)$ by Lemma 1. We want to show

$$\lim_{N\to\infty} S(\Sigma_N)(x_0) = 1/\mathrm{N}(x_0).$$

Since the sets $\Sigma_N$ exhaust $\Delta_\infty$, we have $x_0 \in \Sigma_N$ for sufficiently large $N$. Because $\Sigma_N$ is convex, we know that the singular hyperplanes of the rational function $S(\Sigma_N)(x)$ miss $x_0$. Hence the sequence of partial sums in the limit is well-defined.

The cycle machinery from §§3–7 implies

$$S(\Sigma_N)(x_0) = \tilde{h}(z(\Sigma_N)^*)(x_0)$$

and

$$z(\Sigma_N)^* = z(\Sigma_N^*).$$

Hence we can compute $S(\Sigma_N)(x_0)$ by arbitrarily dividing $\Sigma_N^*$ into *simplicial* cones

$$\Sigma_N^* = \bigcup_{\sigma\in T} \sigma,$$

where $T$ is some finite set of top dimensional cones (not in $C$, of course). We can then apply the relation in $Z(\mathbf{P}^{n-1})$

$$z(\Sigma_N^*) = \sum_{\sigma\in T} z(\sigma).$$

Now we use Hurwitz's interpretation of the function $h$. For any cone $\sigma \subset \mathbf{R}^n$, let $\mathbf{P}\sigma$ be the induced set in $\mathbf{P}^{n-1}$. Let $H_{x_0}$ be the hyperplane in $\mathbf{P}^{n-1}$ determined by the hyperplane in $\mathbf{R}^n$ orthogonal to $x_0$. If $N$ is sufficiently large, then $x_0 \in \Delta_N^*$ and $H_{x_0}$ misses $\mathbf{P}\Delta_N^*$. Thus $H_{x_0}$ eventually misses $\mathbf{P}\Sigma_N^*$ and the sets $\{\mathbf{P}\sigma \mid \sigma \in T\}$. This implies that all three sets $\mathbf{P}\Sigma_N^*$, $\mathbf{P}\Delta_N^*$, and $\mathbf{P}\Delta_\infty$ are contained in the affine chart $\mathbf{P}^{n-1} \setminus H_{x_0}$. Therefore we may fix an orientation such that

$$S(\Sigma_N^*)(x_0) = \int_{\mathbf{P}\Sigma_N^*} \Omega_{x_0},$$

and similarly for $\Delta_\infty$ and $\Delta_N^*$. Since $\Delta_N^* \supset \Sigma_N^* \supset \Delta_\infty$ and $\Delta_N^* \to \Delta_\infty$ as $N \to \infty$ (Theorem 3), we have

$$\int_{\mathbf{P}\Sigma_N^*} \Omega_{x_0} \longrightarrow \int_{\mathbf{P}\Delta_\infty} \Omega_{x_0}.$$

Since $\int_{\mathbf{P}\Delta_\infty} \Omega_{x_0} = 1/\mathrm{N}(x_0)$, Theorem 1 is proved.

## 11. References

[AMRT] A. Ash, D. Mumford, M. Rapoport, and Y. Tai, *Smooth compactifications of locally symmetric varieties*. Math. Sci. Press, Brookline, Mass., 1975.

[ADS] M. Atiyah, H. Donnelley, and I. M. Singer, *Eta invariants, signature defects of cusps, and values of L-functions*. Ann. of Math. **118** (1983), pp. 131–177.

[Ful] W. Fulton, *Introduction to toric varieties*. Ann. of Math. Studies 131, Princeton 1993.

[GP] H. Cohen et al., *The GP-Pari system*, available from `ftp://megrez.math.u-bordeaux.fr/`.

[Hir] F. Hirzebruch, *Hilbert modular surfaces*. Enseignement Math. (2) **19** (1973), 183–281.

[Hur] A. Hurwitz, *Über die Anzahl der Klassen positiver ternärer quadratischer Formen von gegebener Determinante*. Math. Ann. **88** (1923), pp. 26–52.

[KKMS] G. Kempf, F. Knudsen, D. Mumford, and B. St. Donat, *Toroidal embeddings I*. Lecture notes in math. **339** (1973), Springer-Verlag.

[Mey] C. Meyer, *Die Berechnung der Klassenzahl abelischer Körper über quadratischen Zahlkörpern*. Berlin 1957.

[Mül] W. Müller. *Manifolds with cusps of rank one*. Lecture notes in math. **1244** (1987), Springer-Verlag.

[Oga] S. Ogata, *Generalized Hirzebruch's conjecture for Hilbert-Picard modular cusps*. Japan. J. Math. (N.S.) **22** (1996), no. 2, 385–410.

[Sa] I. Satake, *On zeta functions associated with cones and cusp singularities of the second kind*, in Automorphic forms and related topics, Jin-Woo Son and Jae-Hyun Yang, eds., pp. 17–26, Pyungsan Institute for Mathematical Sciences, Seoul, Korea.

[Sc1] R. Sczech, *Eisenstein group cocycles for $GL_n$ and values of L-functions*. Invent. math. **113** (1993), pp. 581–616.

[Sc2] R. Sczech, *Intersection numbers of cusp divisors and values of Hecke L-functions*. Preprint 2001.

[Shi] J.-Y. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*. Lecture notes in math. **1179**, Springer-Verlag.

[Zag] D. Zagier, *Nombres de classes et fractions continues*. Journées Arithmétiques de Bordeaux (Conf., Univ. Bordeaux, Bordeaux, 1974), pp. 81–97. Asterisque, No. 24-25.

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