WEYL GROUP MULTIPLE DIRICHLET SERIES OF TYPE A_2

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Abstract. A *Weyl group multiple Dirichlet series* is a Dirichlet series in several complex variables attached to a root system Φ. The number of variables equals the rank r of the root system, and the series satisfies a group of functional equations isomorphic to the Weyl group W of Φ. In this paper we construct a Weyl group multiple Dirichlet series over the rational function field using n^{th} order Gauss sums attached to the root system of type A_2. The basic technique is that of [10, 11]; namely, we construct a rational function in r variables invariant under a certain action of W, and use this to build a "local factor" of the global series.

In memory of Serge Lang

1. Introduction

Weyl group multiple Dirichlet series are Dirichlet series in r complex variables $s_1, s_2, \ldots, s_r$ that have analytic continuation to $\mathbb{C}^r$, satisfy a group of functional equations isomorphic to the Weyl group of a finite root system of rank r, and whose coefficients are products of n^{th} order Gauss sums. The study of these series was introduced in [2], which also suggested a method for proving their analytic continuation and functional equations.

Recently a complete proof of these expected properties has been given in [12]. In this paper we describe in detail the construction for the root system A_2. There exist alternate constructions of the series defined here. For A_2 and $n \geq 2$ one falls in the stable range, and therefore our result follows from the work of [3]. (In fact, this case was treated earlier in [2].) Nevertheless there are several reasons why a new treatment of A_2 is desirable. First, the methods used here are completely different from those of [3] and give an alternative technique to construct Weyl group multiple Dirichlet series. Second, the technique presented here works for a root system Φ of arbitrary rank and for arbitrary n, with no stability restriction. This is the subject of [12]; one of the main goals of the present paper is an exposition of our method in the simplest nontrivial case, namely $\Phi = A_2$.

With this latter goal in mind we also adopt certain assumptions to make the exposition simpler. For instance, we work over a rational function field to avoid the annoyance of having to deal with Hilbert symbols. We also focus

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on the *untwisted case* (see §2 for an explanation of this terminology) to avoid some notational complexities. A comparison with the methods of [2, 10, 11] indicates how to extend our methods to an arbitrary global field containing the $2n^{th}$ roots of unity and to arbitrary twists.

We now describe our main result in greater detail. Let $\mathbb{F}$ be a finite field whose cardinality q is congruent to 1 mod $4n$. Let K be the rational function field $\mathbb{F}(t)$, and let $\mathcal{O} = \mathbb{F}[t]$. Let $\mathcal{O}_{\mathrm{mon}} \subset \mathcal{O}$ be the subset of monic polynomials. We let $K_\infty = \mathbb{F}[\![t^{-1}]\!]$ denote the field of Laurent series in t^{-1}.

For $x, y \in \mathcal{O}$ relatively prime, we denote by $\left(\frac{x}{y}\right)$ the n^{th} order power residue symbol. We have the reciprocity law

$$\left(\frac{x}{y}\right) = \left(\frac{y}{x}\right) \tag{1.1}$$

for x, y monic. The reciprocity law takes this particularly simple form because of our assumption that the cardinality of $\mathbb{F}$ is congruent to 1 mod 4.

Let $y \mapsto e(y)$ be an additive character on K_∞ with the following property: if $I \subset K$ is the set of all $y \in K$ such that the restriction of e to $y\mathcal{O}$ is trivial, then $I = \mathcal{O}$. Fix an embedding ϵ from the the n^{th} roots of unity in $\mathbb{F}$ to $\mathbb{C}^\times$. For $r, c \in \mathcal{O}$ we define the Gauss sum $g(r, \epsilon, c)$ by

$$g(r, \epsilon, c) = \sum_{y \bmod c} \epsilon\left(\left(\frac{y}{c}\right)\right) e\left(\frac{ry}{c}\right).$$

We will also use the notation $g_i(r, c) = g(r, \epsilon^i, c)$ and $g(r, c) = g(r, \epsilon, c)$. Note that ϵ^i is not necessarily an embedding.

We are now ready to define our double Dirichlet series. Put

$$Z(s_1, s_2) = \tag{1.2}$$
$$(1-q^{n-ns_1})^{-1}(1-q^{n-ns_2})^{-1}(1-q^{2n-ns_1-ns_2})^{-1} \sum_{c_1 \in \mathcal{O}_{\mathrm{mon}}} \sum_{c_2 \in \mathcal{O}_{\mathrm{mon}}} \frac{H(c_1, c_2)}{|c_1|^{s_1}|c_2|^{s_2}},$$

where the coefficient $H(c_1, c_2)$ is defined as follows:

(1) (Twisted multiplicativity) If $gcd(c_1c_2, d_1d_2) = 1$ then

$$\frac{H(c_1d_1, c_2d_2)}{H(c_1, c_2)H(d_1, d_2)} = \left(\frac{c_1}{d_1}\right)\left(\frac{d_1}{c_1}\right)\left(\frac{c_2}{d_2}\right)\left(\frac{d_2}{c_2}\right)\left(\frac{c_1}{d_2}\right)^{-1}\left(\frac{d_1}{c_2}\right)^{-1}. \tag{1.3}$$

(2) ($\mathfrak{p}$-part) If $\mathfrak{p}$ is prime, then

$$\sum_{k,l \geq 0} H(\mathfrak{p}^k, \mathfrak{p}^l)x^k y^l = \tag{1.4}$$
$$1 + g(1,\mathfrak{p})x + g(1,\mathfrak{p})y + g(1,\mathfrak{p})g(\mathfrak{p},\mathfrak{p})xy + g(1,\mathfrak{p})g(\mathfrak{p},\mathfrak{p}^2)xy^2$$
$$+ g(1,\mathfrak{p})g(\mathfrak{p},\mathfrak{p}^2)x^2y + g(1,\mathfrak{p})^2 g(\mathfrak{p},\mathfrak{p}^2)x^2y^2.$$

Our main result is

Theorem 1.1. *The double Dirichlet series $Z(s_1, s_2)$ converges absolutely for $Re(s_i)$ sufficiently large and has an analytic continuation to all $(s_1, s_2) \in \mathbb{C}^2$. Moreover, $Z(s_1, s_2)$ satisfies two functional equations of the form*

$$(1.5)\quad \sigma_1\colon (s_1, s_2) \mapsto (2-s_1, s_1+s_2-1) \text{ and } \sigma_2\colon (s_1, s_2) \mapsto (s_1+s_2-1, 2-s_2).$$

These two functional equations generate a subgroup of the affine transformations of $\mathbb{C}^2$ isomorphic to the symmetric group S_3.

The precise statement of the functional equations involves a set of double Dirichlet series $Z(s_1, s_2; i, j)$, where $0 \leq i, j \leq n-1$, and where $Z(s_1, s_2) = \sum_{i,j} Z(s_1, s_2; i, j)$; we refer to Theorem 4.1 for details. Moreover, one can explicitly write down $Z(s_1, s_2)$ as a rational function in q^{-s_1}, q^{-s_2}. For $n = 2$, this was first done by Hoffstein and Rosen [16], and later by Fisher and Friedberg [13], whose approach is closer to the point of view of this paper. For $n > 2$ the A_2 series have been computed by Chinta [8].

As stated above this theorem follows from the work of [2,3]. In [6], the authors study the harder problem of constructing twisted Weyl group multiple Dirichlet series associated to the root system A_r. They construct such series for A_2 and present a conjectural description of the series associated to A_r for arbitrary r and n. Recently, Brubaker, Bump and Friedberg have given two different proofs of their conjectures [4, 5], thereby giving a complete definition of Weyl group multiple Dirichlet series associated to A_r.

Our method has the advantage that functional equations are essentially built-in to our definition. As in the case of [2, 3, 6, 10, 11] the Weyl group multiple Dirichlet series are completely determined by their $\mathfrak{p}$-parts and the twisted multiplicativity satisfied by the coefficients. Our approach is to show that if the $\mathfrak{p}$-parts (which can be expressed as rational functions in the $|\mathfrak{p}|^{-s_i}$) satisfy certain functional equations, then the global multiple Dirichlet series satisfies the requisite global functional equations. This leads us to define a certain action of W, the Weyl group of the root system Φ, on a certain subring of the field of rational functions in r indeterminates. This approach, first introduced in [7], has been carried out in the quadratic case for an arbitrary simply-laced root system, see [10, 11]. We extend this approach to arbitrary Φ and n in [12]. However, though the basic ideas are clear, the non-obvious group action required on rational functions can appear unmotivated and complicated in the general setting. Therefore, we feel it is worthwhile in this paper to work out in detail the simplest nontrivial case, the rank two root system A_2.

Here is a short plan of the paper. Section 2 describes the Weyl group action on rational functions that leads to a $\mathfrak{p}$-part (1.4) with the desired functional equations. Although the focus of this paper is untwisted A_2, we work more generally at first and state the full action for a general (simply laced) root system. We then specialize to untwisted A_2. Section 3 reviews the Dirichlet series of Kubota; in the current framework, these series are Weyl group multiple Dirichlet series attached to A_1. The main result of this section is Theorem 3.4, which shows that a certain Dirichlet series $E(s, m)$

built from the function $H(c,d)$ from (1.3)–(1.4) satisfies the same functional equations as Kubota's. Finally, in Section 4 we use Theorem 3.4 to complete the proof of Theorem 1.1. The basic idea is that the (one variable) functional equations of the $E(s,m)$ induce a bivariate functional equation in the double Dirichlet series.

2. A Weyl group action

Let Φ be an irreducible simply laced root system of rank r with Weyl group W. Choose an ordering of the roots and let $\Phi = \Phi^+ \cup \Phi^-$ be the decomposition into positive and negative roots. Let

$$\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$$

be the set of simple roots and let σ_i be the Weyl group element corresponding to the reflection through the hyperplane perpendicular to α_i. We say that i and j are *adjacent* if $i \neq j$ and $(\sigma_i\sigma_j)^3 = 1$. The Weyl group W is generated by the simple reflections $\sigma_1, \sigma_2, \dots, \sigma_r$, which satisfy the relations

$$(\sigma_i\sigma_j)^{r(i,j)} = 1 \text{ with } r(i,j) = \begin{cases} 3 & \text{if } i \text{ and } j \text{ are adjacent,} \\ 1 & \text{if } i = j, \text{ and} \\ 2 & \text{otherwise,} \end{cases} \tag{2.1}$$

for $1 \le i, j \le r$. The action of the generators σ_i on the roots is

$$\sigma_i\alpha_j = \begin{cases} \alpha_i + \alpha_j & \text{if } i \text{ and } j \text{ are adjacent,} \\ -\alpha_j & \text{if } i = j, \text{ and} \\ \alpha_j & \text{otherwise.} \end{cases} \tag{2.2}$$

Define

$$\operatorname{sgn}(w) = (-1)^{length(w)}$$

where the length function on W is with respect to the generators $\sigma_1, \sigma_2, \dots, \sigma_r$. Let Λ be the lattice generated by the roots. Any $\alpha \in \Lambda$ has a unique representation as an integral linear combination of the simple roots:

$$\alpha = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_r\alpha_r. \tag{2.3}$$

We denote by

$$d(\alpha) = k_1 + k_2 + \cdots + k_r$$

the usual height function on Λ and put

$$d_j(\alpha) = \sum_{i\sim j} k_i,$$

where $i \sim j$ means that the nodes labeled by i and j are adjacent in the Dynkin diagram of Φ. Introduce a partial ordering on Λ by defining $\alpha \succeq 0$ if each $k_i \geq 0$ in (2.3). Given $\alpha, \beta \in \Lambda$, define $\alpha \succeq \beta$ if $\alpha - \beta \succeq 0$.

Let $A = \mathbb{C}[\Lambda]$ be the ring of Laurent polynomials on the lattice Λ. Hence A consists of all expressions of the form $f = \sum_{\beta\in\Lambda} a_\beta \mathbf{x}^\beta$, where $a_\beta \in \mathbb{C}$ and almost all are zero, and the multiplication of monomials is defined by addition in Λ: $\mathbf{x}^\beta\mathbf{x}^\lambda = \mathbf{x}^{\beta+\lambda}$. We identify A with $\mathbb{C}[x_1, x_1^{-1}, \dots, x_r, x_r^{-1}]$ via $\mathbf{x}^{\alpha_i} \mapsto x_i$.

Let $\mathfrak{p}$ be a prime in $\mathcal{O}$ of norm p. Let $\widetilde{A}$ be the localization of A at the multiplicative subset of all expressions of the form

$$\{1 - p^{nd(\alpha)}\mathbf{x}^{nd(\alpha)}, 1 - p^{nd(\alpha)-1}\mathbf{x}^{nd(\alpha)} \mid \alpha \in \Phi^+\}.$$

The group W will act on $\widetilde{A}$, and the action will involve the Gauss sums $g_i(1, \mathfrak{p})$.[1] There is one further parameter necessary for the definition. Let $\ell = (l_1, \ldots, l_r)$ be an r-tuple of nonnegative integers. The tuple ℓ is called a *twisting parameter*; it should be thought of as corresponding to the weight $\sum(l_j + 1)\varpi_j$, where the ϖ_j are the fundamental weights of Φ. The case $\ell = (0, \ldots, 0)$ is called the *untwisted case*. For each choice of ℓ we will define an action of the Weyl group W on $\widetilde{A}$.

We are now ready to define the W-action. First, we define a "change of variables" action on $\widetilde{A}$ as follows. for $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ define $\sigma_i \mathbf{x} = \mathbf{x}'$, where

$$x'_j = \begin{cases} px_ix_j & \text{if } i \text{ and } j \text{ are adjacent,} \\ 1/(p^2x_j) & \text{if } i = j, \text{ and} \\ x_j & \text{otherwise.} \end{cases} \tag{2.4}$$

One can easily check that if $f_\beta(\mathbf{x}) = \mathbf{x}^\beta$ is a monomial, then

$$f_\beta(w\mathbf{x}) = q^{d(w^{-1}\beta - \beta)}\mathbf{x}^{w^{-1}\beta}. \tag{2.5}$$

Next, write $f \in A$ as

$$f(\mathbf{x}) = \sum_\beta a_\beta \mathbf{x}^\beta.$$

Given integers k, i, j, define

$$f_k(\mathbf{x}; i, j) = \sum_{\substack{\beta_k = i \bmod n \\ d_k(\beta) = j \bmod n}} a_\beta \mathbf{x}^\beta.$$

We define the action of a generator $\sigma_k \in W$ on f as follows:

$$(f|_\ell \sigma_k)(\mathbf{x}) = \tag{2.6}$$

$$(px_k)^{l_k} \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} (\mathcal{P}_{ij}(x_k) f_k(\sigma_k\mathbf{x}; i, j - l_k) + \mathcal{Q}_{ij}(x_k) f_k(\sigma_k\mathbf{x}; j + 1 - i, j - l_k))$$

where

$$\begin{aligned} \mathcal{P}_{ij}(x) &= (px)^{1-(-2i+j+1)_n} \frac{1 - 1/p}{1 - p^{n-1}x^n}, \\ \mathcal{Q}_{ij}(x) &= -g^*_{2i-j-1}(1, \mathfrak{p})(px)^{1-n} \frac{1 - p^n x^n}{1 - p^{n-1}x^n}, \\ g^*_i(1, \mathfrak{p}) &= \begin{cases} g_i(1, \mathfrak{p})/p & \text{if } n \nmid i, \\ -1 & \text{otherwise.} \end{cases} \end{aligned}$$

[1] We remark that our normalization for Gauss sums follows [3, 6] and not [10, 11]. See [11, Remark 3.12] for a discussion of this.

Here $(i)_n \in \{0, \dots, n-1\}$ is the remainder upon division of i by n. We extend this action to all of $\widetilde{A}$ first extending (2.6) to all of A by linearity, and then given $f/g \in \widetilde{A}$ by defining

$$\left(\frac{f}{g}\Big|_\ell \sigma_k\right)(\mathbf{x}) = \frac{(f|_\ell\sigma_k)(\mathbf{x})}{g(\sigma_k\mathbf{x})}.$$

One can show that this action of the generators extends to an action of W on $\widetilde{A}$; in particular the defining relations (2.1) are satisfied.

Now we specialize to the focus of this paper: we set $\Phi = A_2$ and $\ell = (0,0)$. To simplify notation we write x, y for the variables of $\widetilde{A}$. With these simplifications the action of σ_1 on $f \in A$ takes the form

$$(f|\sigma_1)(x,y) = \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\left(\mathcal{P}_{ij}(x) f_1\left(\tfrac{1}{p^2x}, pxy; i, j\right) + \mathcal{Q}_{ij}(x) f_1\left(\tfrac{1}{p^2x}, pxy; j+1-i, j\right)\right); \tag{2.7}$$

the action of σ_2 is similar. An invariant rational function for this action is

$$h(x,y) = \frac{N(x,y)}{(1-p^{n-1}x^n)(1-p^{n-1}y^n)(1-p^{2n-1}x^ny^n)}, \tag{2.8}$$

where the numerator $N(x,y)$ is

$$\begin{aligned} N(x,y) = N^{(\mathfrak{p})}(x,y) = {}& \\ 1 + g_1(1,\mathfrak{p})x + g_1(1,\mathfrak{p})y + g_1(1,\mathfrak{p})g_1(\mathfrak{p},\mathfrak{p})xy + pg_1(1,\mathfrak{p})g_2(1,\mathfrak{p})xy^2 + {}& \\ pg_1(1,\mathfrak{p})g_2(1,\mathfrak{p})x^2y + pg_1(1,\mathfrak{p})^2g_2(1,\mathfrak{p})x^2y^2.& \end{aligned} \tag{2.9}$$

To compare this with (1.4), note that $pg_2(1,\mathfrak{p}) = g_1(\mathfrak{p},\mathfrak{p}^2)$. Also note that only the numerator of (2.8) appears in (1.4) because the denominator is incorporated in the factors appearing at the front of (1.2).

Let us write $h(x,y)$ as

$$\begin{aligned} h(x,y) &= \sum_{k,l\geq 0} a(\mathfrak{p}^k,\mathfrak{p}^l)x^ky^l \\ &= \sum_{l\geq 0} y^l \left(\sum_{i=0}^{n-1}\sum_{k=i \bmod n} a(\mathfrak{p}^k,\mathfrak{p}^l)x^k\right) \\ &= \sum_{l\geq 0}\sum_{i=0}^{n-1} y^l h^{(\mathfrak{p},l)}(x;i), \end{aligned} \tag{2.10}$$

say.

The following two lemmas are proved by a direct computation.

Lemma 2.1. *We have* $N^{(\mathfrak{p})}(x,0) = 1 + g_1(1,\mathfrak{p})x$, $N^{(\mathfrak{p})}(0,y) = 1 + g_1(1,\mathfrak{p})y$ *and for* $j = l \bmod n$, *and* $0 \leq i \leq n-1$,

$$h^{(\mathfrak{p},l)}(x;i) = (px)^l P_{ij}(x) h\left(\tfrac{1}{p^2x}; i\right) + (px)^l Q_{ij}(x) h\left(\tfrac{1}{p^2x}; l+1-i\right).$$

Lemma 2.2. *Let*
$f^{(\mathfrak{p},l)}(x;i) = h^{(\mathfrak{p},l)}(x;i) - \delta g_{2i-l-1}(1,\mathfrak{p})p^{(2i-l-2)_n}x^{(2i-l-1)_n}h^{(\mathfrak{p},l)}(x,l+1-i)$
where $\delta = 0$ *if* $l-2i = -1 \bmod n$ *and is 1 otherwise. Then*

$$f^{(\mathfrak{p},l)}(x;i) = (px)^{l-(l-2i)_n} f^{(\mathfrak{p},l)}\left(\tfrac{1}{p^2x};i\right).$$

3. Kubota's Dirichlet series

The basic building blocks of the multiple Dirichlet series are the Kubota Dirichlet series constructed from Gauss sums [17, 18]. Let m be a nonzero polynomial in $\mathcal{O}$ and let s be a complex variable. These series are defined by

$$D(s,m) = (1-q^{n-ns})^{-1} \sum_{d \in \mathcal{O}_{\mathrm{mon}}} \frac{g(m,d)}{|d|^s} \tag{3.1}$$

and

$$D(s,m;i) = (1-q^{n-ns})^{-1} \sum_{\substack{\deg d = i \bmod n \\ d \in \mathcal{O}_{\mathrm{mon}}}} \frac{g(m,d)}{|d|^s}. \tag{3.2}$$

Kubota proved that these series have meromorphic continuation to $s \in \mathbb{C}$ with possible poles only at $s = 1 \pm 1/n$ and satisfy a functional equation. Actually, Kubota worked over a number field, but the constructions over a function field are identical.

If the degree of m is $nk+j$, where $0 \le j \le n-1$, this functional equation takes the form

$$D(s,m) = |m|^{1-s} \sum_{0 \le i \le n-1} T_{ij}(s) D(2-s,m;i), \tag{3.3}$$

where the $T_{ij}(s)$ are certain quotients of Dirichlet polynomials. For fixed s the T_{ij} depend only on $2i-j$. We will not need to know anything more about the functional equation, but a more explicit description can be found in Hoffstein [15] or Patterson [20].

Given a set of primes S, we define

$$D_S(s,m) = (1-q^{n-ns})^{-1} \sum_{\substack{(d,S)=1 \\ d \in \mathcal{O}_{\mathrm{mon}}}} \frac{g(m,d)}{|d|^s}. \tag{3.4}$$

If $m_0 = \prod_{\mathfrak{p} \in S} \mathfrak{p}$ we sometimes write $D_{m_0}(s,m)$ for $D_S(s,m)$.

We record some properties of Gauss sums that we will use repeatedly.

Proposition 3.1. *Let* $a, m, c, c' \in \mathcal{O}$.

(i) If $(a,c) = 1$ *then* $g_i(am,c) = \left(\frac{a}{c}\right)^{-1} g_i(m,c)$.

(ii) If $(c,c') = 1$ *then*

$$g_i(m,cc') = g_i(m,c)g_i(m,c')\left(\frac{c}{c'}\right)^{2i}.$$

Using this proposition we can relate the functions D_S to the functions $D_{S'}$ for different sets S and S'. This is the content of the following two lemmas.

Lemma 3.2. *Let $\mathfrak{p} \in \mathcal{O}_{mon}$ be prime of norm p. For an integer i with $0 \leq i \leq n-1$ and $m_1, m_2, \mathfrak{p}$ all pairwise relatively prime, we have*

$$D_{m_1}(s, m_2\mathfrak{p}^i) = D_{\mathfrak{p}m_1}(s, m_2\mathfrak{p}^i) + \frac{g(m_2\mathfrak{p}^i, \mathfrak{p}^{i+1})}{p^{(i+1)s}} D_{\mathfrak{p}m_1}(s, m_2\mathfrak{p}^{(n-i-2)_n}).$$

More generally,

$$D(s,m) = \sum_{S_0 \subset S} \left(\prod_{\mathfrak{p} \in S_0} \frac{g(m, \mathfrak{p}^{i+1})}{|\mathfrak{p}|^{(i+1)s}} \right) D_S \left(s, \prod_{\substack{\mathfrak{p} \in S_0^c \\ \mathfrak{p}^i || m}} \mathfrak{p}^i \cdot \prod_{\substack{\mathfrak{p} \in S_0 \\ \mathfrak{p}^i || m}} \mathfrak{p}^{(n-i-2)_n} \right).$$

Proof. We prove only the first part of the Lemma. For $\mathfrak{p}, m_1, m_2$ as in the statement,

$$\begin{aligned}
(1 - q^{n-ns}) D_{m_1}(s, m_2\mathfrak{p}^i) &= \sum_{\substack{(d, m_1)=1 \\ d \in \mathcal{O}_{\mathrm{mon}}}} \frac{g(m_2\mathfrak{p}^i, d)}{|d|^s} \\
&= \sum_{k \geq 0} \sum_{\substack{(d, m_1\mathfrak{p})=1 \\ d \in \mathcal{O}_{\mathrm{mon}}}} \frac{g(m_2\mathfrak{p}^i, d\mathfrak{p}^k)}{|d|^s p^{ks}} \\
&= \sum_{k \geq 0} \sum_{\substack{(d, m_1\mathfrak{p})=1 \\ d \in \mathcal{O}_{\mathrm{mon}}}} \frac{g(m_2\mathfrak{p}^i, d) g(m_2\mathfrak{p}^i, \mathfrak{p}^k)}{|d|^s p^{ks}} \left(\frac{d}{\mathfrak{p}^{2k}} \right) \\
&= \sum_{\substack{(d, m_1\mathfrak{p})=1 \\ d \in \mathcal{O}_{\mathrm{mon}}}} \frac{g(m_2\mathfrak{p}^i, d)}{|d|^s} \left(\sum_{k \geq 0} \frac{g(m_2\mathfrak{p}^i, \mathfrak{p}^k)}{p^{ks}} \left(\frac{d}{\mathfrak{p}^{2k}} \right) \right).
\end{aligned}$$

The Gauss sum in the inner sum vanishes unless $k = 0$ or $i + 1$. This proves the Lemma. ☐

Inverting the previous Lemma, we obtain

Lemma 3.3. *If $0 \leq i \leq n-2$ and $m_1, m_2, \mathfrak{p}$ as above,*

$$D_{\mathfrak{p}m_1}(s, m_2\mathfrak{p}^i) = \frac{D_{m_1}(s, m_2\mathfrak{p}^i)}{1 - |\mathfrak{p}|^{n-1-ns}} - \frac{g(m_2\mathfrak{p}^i, \mathfrak{p}^{i+1})}{|\mathfrak{p}|^{(i+1)s}} \frac{D_{m_1}(s, m_2\mathfrak{p}^{n-i-2})}{1 - |\mathfrak{p}|^{n-1-ns}},$$

and if $i = n-1$,

$$D_{\mathfrak{p}m_1}(s, m_2\mathfrak{p}^i) = \frac{D_{m_1}(s, m_2\mathfrak{p}^i)}{1 - |\mathfrak{p}|^{n-1-ns}}.$$

Now suppose that $N(x,y) = N^{(\mathfrak{p})}(x,y)$ is the polynomial from (2.9). We define a function H on pairs of powers of $\mathfrak{p}$ by setting $H(\mathfrak{p}^k,\mathfrak{p}^l)$ to be the coefficient of $x^k y^l$ in $N(x,y)$:

$$N(x,y) = \sum H(\mathfrak{p}^k,\mathfrak{p}^l)x^k y^l.$$

We extend H to all pairs of monic polynomials by the twisted multiplicativity relation: if $\gcd(cd, c'd') = 1$, then we put

$$H(cc', dd') = H(c,d)H(c',d')\Big(\frac{c}{c'}\Big)^2\Big(\frac{d}{d'}\Big)^2\Big(\frac{c}{d'}\Big)^{-1}\Big(\frac{c'}{d}\Big)^{-1}. \tag{3.5}$$

In particular, note that

$$H(d,1) = g(1,d). \tag{3.6}$$

Now consider the Dirichlet series

$$E(s,m) = (1-q^{n-ns})^{-1}\sum_{d\in\mathcal{O}_{\mathrm{mon}}}\frac{H(d,m)}{d^s}. \tag{3.7}$$

That $E(s,m)$ satisfies the same functional equation as $D(s,m)$ is the main result of this section:

Theorem 3.4. *Let $m \in \mathcal{O}_{mon}$ be a monic polynomial of degree $nk+j$, where $0 \le j \le n-1$. Then*

$$E(s,m) = |m|^{1-s}\sum_{0\le i\le n-1} T_{ij}(s)E(2-s,m;i).$$

Proof. Before tackling the general case, we first consider $m = \mathfrak{p}^l$ for a prime $\mathfrak{p}$. Then

$$\begin{aligned}
E(s,\mathfrak{p}^l) &= (1-q^{n-ns})^{-1}\sum_{\substack{d\in\mathcal{O}_{\mathrm{mon}}\\(d,\mathfrak{p})=1}}\sum_{k\ge0}\frac{H(d\mathfrak{p}^k,\mathfrak{p}^l)}{d^s|\mathfrak{p}|^{ks}}\\
&= (1-q^{n-ns})^{-1}\sum_{\substack{d\in\mathcal{O}_{\mathrm{mon}}\\(d,\mathfrak{p})=1}}\sum_{k\ge0}\frac{H(\mathfrak{p}^k,\mathfrak{p}^l)g(1,d)}{|\mathfrak{p}|^{ks}d^s}\Big(\frac{d}{\mathfrak{p}^{2k-l}}\Big), \text{ by (3.5) and (3.6)}\\
&= \sum_{k\ge0}\frac{H(\mathfrak{p}^k,\mathfrak{p}^l)}{|\mathfrak{p}|^{ks}}D_{\mathfrak{p}}(s,\mathfrak{p}^{(l-2k)_n})\\
&= \sum_{j=0}^{n-1}D_{\mathfrak{p}}(s,\mathfrak{p}^{(l-2j)_n})\left(\frac{1}{|\mathfrak{p}|^{js}}\sum_{k\ge0}\frac{H(\mathfrak{p}^{j+nk},\mathfrak{p}^l)}{|\mathfrak{p}|^{nks}}\right)\\
&= \sum_{j=0}^{n-1}D_{\mathfrak{p}}(s,\mathfrak{p}^{(l-2j)_n})h^{(\mathfrak{p},l)}(|\mathfrak{p}|^{-s};j),
\end{aligned}$$

where $h^{(\mathfrak{p},l)}$ was introduced in (2.10). Using Lemma 3.3 the previous expression becomes

$$\begin{aligned}(3.8)\qquad &\sum_{j=0}^{n-1} D(s,\mathfrak{p}^{(l-2j)_n})h^{(\mathfrak{p},l)}(|\mathfrak{p}|^{-s};j)\\ &-\sum_{j=0}^{n-1}\delta_j\frac{g(\mathfrak{p}^{(l-2j)_n},\mathfrak{p}^{(l-2j)_n+1})}{|\mathfrak{p}|^{((l-2j)_n+1)s}}D(s,\mathfrak{p}^{(2j-l-2)_n})h^{(\mathfrak{p},l)}(|\mathfrak{p}|^{-s};j),\end{aligned}$$

where $\delta_j = 0$ if $l-2j \equiv n-1(n)$ and is 1 otherwise. Replace j by $l+1-j$ in the second summation and regroup to conclude

$$E(s,\mathfrak{p}^l) = \sum_{j=0}^{n-1} D(s,\mathfrak{p}^{(l-2j)_n})f^{(\mathfrak{p},l)}(|\mathfrak{p}|^{-s};j). \tag{3.9}$$

(Note the use of the identity $n-2-(l-2j)_n = (2j-l-2)_n$.) Using the functional equations (3.3) of D and $f^{(\mathfrak{p},l)}$ (Lemma 2.2), we write

(3.10)
$$\begin{aligned}&E(s,\mathfrak{p}^l)|\mathfrak{p}|^{-(1-s)l}\\ &=\sum_{j=0}^{n-1}\sum_{i=0}^{n-1}T_{i,(l-2j)_n\deg\mathfrak{p}}(s)D(2-s,\mathfrak{p}^{(l-2j)_n};i)f^{(\mathfrak{p},l)}(2-s;j)\\ &=\sum_{i,j=0}^{n-1}T_{i-j\deg\mathfrak{p},(l-2j)_n\deg\mathfrak{p}}(s)D(2-s,\mathfrak{p}^{(l-2j)_n};i-j\deg\mathfrak{p})f^{(\mathfrak{p},l)}(2-s;j)\\ &=\sum_{i=0}^{n-1}T_{i,l\deg\mathfrak{p}}(s)\left[\sum_{j=0}^{n-1}D(2-s,\mathfrak{p}^{(l-2j)_n};i-j\deg\mathfrak{p})f^{(\mathfrak{p},l)}(2-s;j)\right]\\ &=\sum_{i=0}^{n-1}T_{i,l\deg\mathfrak{p}}(s)E(2-s,\mathfrak{p}^l;i),\end{aligned}$$

where the third equality comes from our remark that the T_{ij} depend only on $2i-j$. This is the functional equation we wished to prove, in the special case $m=\mathfrak{p}^l$.

The argument for general m is similar. Let $m = \mathfrak{p}_1^{l_1}\mathfrak{p}_2^{l_2}\cdots\mathfrak{p}_r^{l_r}$ where the $\mathfrak{p}_i$ are distinct primes. Then

(3.11)
$$\begin{aligned}
E(s;m) &= (1-q^{n-ns})^{-1}\sum_{d\in\mathcal{O}_{\mathrm{mon}}}\frac{H(d,m)}{|d|^s}\\
&= (1-q^{n-ns})^{-1}\sum_{\substack{d\in\mathcal{O}_{\mathrm{mon}}\\(d,m)=1}}\sum_{k_1,\ldots,k_r\ge 0}\frac{H(d\mathfrak{p}_1^{k_1}\cdots\mathfrak{p}_r^{k_r},\mathfrak{p}_1^{l_1}\cdots\mathfrak{p}_r^{l_r})}{|d|^s|\mathfrak{p}_1|^{k_1 s}\cdots|\mathfrak{p}_r|^{k_r s}}\\
&= (1-q^{n-ns})^{-1}\sum_{\substack{d\in\mathcal{O}_{\mathrm{mon}}\\(d,m)=1}}\sum_{k_1,\ldots,k_r\ge 0}\frac{H(d,1)H(\mathfrak{p}_1^{k_1},\mathfrak{p}_1^{l_1})\cdots H(\mathfrak{p}_r^{k_r},\mathfrak{p}_r^{l_r})}{|d|^s|\mathfrak{p}_1|^{k_1 s}\cdots|\mathfrak{p}_r|^{k_r s}}\\
&\quad\times\left(\frac{d}{m}\right)^{-1}\left(\frac{d}{\mathfrak{p}_1^{k_1}\cdots\mathfrak{p}_r^{k_r}}\right)^2\prod_{a\neq b}\left(\frac{\mathfrak{p}_a^{k_a}}{\mathfrak{p}_b^{k_b}}\right)\left(\frac{\mathfrak{p}_a^{l_a}}{\mathfrak{p}_b^{l_b}}\right)\left(\frac{\mathfrak{p}_a^{k_a}}{\mathfrak{p}_b^{l_b}}\right)^{-1}\\
&= \prod_{a\neq b}\left(\frac{\mathfrak{p}_a^{l_a}}{\mathfrak{p}_b^{l_b}}\right)\sum_{j_1=0}^{n-1}\cdots\sum_{j_r=0}^{n-1}D_m(s,\mathfrak{p}_1^{(l_1-2j_1)_n}\cdots\mathfrak{p}_r^{(l_r-2j_r)_n})\\
&\quad\times h^{(\mathfrak{p}_1,l_1)}(s;j_1)\cdots h^{(\mathfrak{p}_r,l_r)}(s;j_r)\prod_{a\neq b}\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{j_b}}\right)\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{l_b}}\right)^{-1}.
\end{aligned}$$

Denote for the moment by $C(j_1) = C(j_1,\ldots,j_r)$ the product of residue symbols

$$C(j_1) = \prod_{a\neq b}\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{j_b}}\right)\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{l_b}}\right)^{-1}. \tag{3.12}$$

Letting $J_i = (l_i - 2j_i)_n$ for $i = 1,\ldots r$, we have

(3.13)
$$\begin{aligned}
(1-|\mathfrak{p}_1|^{n-1-ns})D_m(s,\mathfrak{p}_1^{J_1}\cdots\mathfrak{p}_r^{J_r})C(j_1) &= D_{\mathfrak{p}_2\cdots\mathfrak{p}_r}(s,\mathfrak{p}_1^{J_1}\cdots\mathfrak{p}_r^{J_r})C(j_1)\\
&\quad-\delta_{j_1}\frac{g(\mathfrak{p}_1^{J_1}\cdots\mathfrak{p}_r^{J_r},\mathfrak{p}_1^{J_1+1})}{|\mathfrak{p}_1|^{(J_1+1)s}}D_{\mathfrak{p}_2\cdots\mathfrak{p}_r}(s,\mathfrak{p}_1^{(2j_1-l_1-2)_n}\mathfrak{p}_2^{J_2}\cdots\mathfrak{p}_r^{J_r})C(j_1)
\end{aligned}$$

by Lemma 3.3. In the second term on the right hand side, replace j_1 by l_1+1-j_1. For $\delta_{j_1}\neq 0$ this gives

(3.14)
$$\frac{g(\mathfrak{p}_1^{(2j_1-l_1-2)_n}\mathfrak{p}_2^{J_2}\cdots\mathfrak{p}_r^{J_r},\mathfrak{p}_1^{(2j_1-l_1-1)_n})}{|\mathfrak{p}_1|^{((2j_1-l_1-1)_n)s}}D_{\mathfrak{p}_2\cdots\mathfrak{p}_r}(s,\mathfrak{p}_1^{J_1}\mathfrak{p}_2^{J_2}\cdots\mathfrak{p}_r^{J_r})C(l_1-j_1+1).$$

The Gauss sum can be written as

$$\left(\frac{\mathfrak{p}_2^{J_2}\cdots\mathfrak{p}_r^{J_r}}{\mathfrak{p}_1^{2j_1-l_1-1}}\right)^{-1}g(\mathfrak{p}_1^{(2j_1-l_1-2)_n},\mathfrak{p}_1^{(2j_1-l_1-1)_n}), \tag{3.15}$$

and $C(l_1 - j_1 + 1)$ is

$$\left(\frac{\mathfrak{p}_2^{J_2}\cdots\mathfrak{p}_r^{J_r}}{\mathfrak{p}_1^{l_1-j_1+1}}\right)^{-1}\left(\frac{\mathfrak{p}_2^{j_2}\cdots\mathfrak{p}_r^{j_r}}{\mathfrak{p}_1^{l_1}}\right)^{-1}\left(\prod_{\substack{a\neq b\\ a,b\neq 1}}\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{j_b}}\right)\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{l_b}}\right)^{-1}\right). \tag{3.16}$$

Taking the product of (3.15) and (3.16) yields

$$\left(\frac{\mathfrak{p}_2^{J_2}\cdots\mathfrak{p}_r^{J_r}}{\mathfrak{p}_1^{j_1}}\right)^{-1}\left(\frac{\mathfrak{p}_2^{j_2}\cdots\mathfrak{p}_r^{j_r}}{\mathfrak{p}_1^{l_1}}\right)^{-1}g(\mathfrak{p}_1^{(2j_1-l_1-2)_n},\mathfrak{p}_1^{(2j_1-l_1-1)_n})\left(\prod_{\substack{a\neq b\\ a,b\neq 1}}\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{j_b}}\right)\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{l_b}}\right)^{-1}\right)$$
$$= g(\mathfrak{p}_1^{(2j_1-l_1-2)_n},\mathfrak{p}_1^{(2j_1-l_1-1)_n})C(j_1). \tag{3.17}$$

Therefore, continuing from the last line of (3.11),

$$\begin{aligned}E(s,m) = {} & \prod_{a\neq b}\left(\frac{\mathfrak{p}_a^{l_a}}{\mathfrak{p}_b^{l_b}}\right)\sum_{j_1=0}^{n-1}\cdots\sum_{j_r=0}^{n-1}D_{m'}(s,\mathfrak{p}_1^{(l_1-2j_1)_n}\cdots\mathfrak{p}_r^{(l_r-2j_r)_n})\\ & \times\prod_{a\neq b}\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{j_b}}\right)\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{l_b}}\right)^{-1}f^{(\mathfrak{p}_1,l_1)}(s;j_1)h^{(\mathfrak{p}_2,l_2)}(s;j_2)\cdots h^{(\mathfrak{p}_r,l_r)}(s;j_r),\end{aligned} \tag{3.18}$$

where $m' = \mathfrak{p}_2^{l_2}\cdots\mathfrak{p}_r^{l_r}$. Repeating this procedure to remove the primes from m one at a time, we find that up to a constant of modulus one, $E(s,m)$ is equal to

$$\sum_{j_1=0}^{n-1}\cdots\sum_{j_r=0}^{n-1}D(s,\mathfrak{p}_1^{J_1}\cdots\mathfrak{p}_r^{J_r})\left(\prod_{a=1}^{r}f^{(\mathfrak{p}_a,l_a)}(s;j_a)\right)\prod_{a\neq b}\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{j_b}}\right)\left(\frac{\mathfrak{p}_a^{j_a}}{\mathfrak{p}_b^{l_b}}\right)^{-1}. \tag{3.19}$$

We may now apply the functional equations of D and the $f^{(\mathfrak{p}_a,l_a)}$ as in (3.10) to conclude that $E(s,m)$ satisfies the functional equation

$$E(s,m) = |m|^{1-s}\sum_{i=0}^{n-1}T_{i,\deg m}(s)E(2-s,m;i). \tag{3.20}$$

This completes the proof of the theorem. □

For later use, we record the following bound:

Proposition 3.5. *For all* $\epsilon > 0$, $m \in \mathcal{O}$ *and* $0 \leq i < n$,

$$(s-1-\tfrac{1}{n})(s-1+\tfrac{1}{n})E(s,m;i) \ll_\epsilon \begin{cases} 1 & \text{for } Re(s) > \frac{3}{2}+\epsilon\\ |m|^{\frac{1}{2}+\epsilon} & \text{for } \frac{1}{2}-\epsilon < Re(s) < \frac{3}{2}+\epsilon\\ |m|^{1-s+\epsilon} & \text{for } Re(s) < \frac{1}{2}-\epsilon\end{cases}$$

Proof. Use the meromorphy and functional equation of $E(s,m)$ together with the convexity principle, cf. [14, Eq. (2.3)] and [19, Propostion 8.4]. □

4. The double Dirichlet series

Recall the definition of the double Dirichlet series from (1.2)–(1.4). In this section we show that $Z(s_1, s_2)$ has a meromorphic continuation to $s_1, s_2 \in \mathbb{C}$ and satisfies a group of functional equation isomorphic to W. In [2], the authors show in detail how the analytic continuation of a Weyl group multiple Dirichlet series follows from the functional equations. Therefore we concentrate on establishing the functional equations of $Z(s_1, s_2)$.

Actually we need to consider slightly different series. For integers $0 \leq i, j \leq n-1$ we define

$$Z(s_1, s_2; i, j) = (1-q^{n-ns_1})^{-1}(1-q^{n-ns_2})^{-1}(1-q^{2n-ns_1-ns_2})^{-1} \sum_{\substack{m\in\mathcal{O}_{\mathrm{mon}}\\ \deg m = i \bmod n}} \sum_{\substack{d\in\mathcal{O}_{\mathrm{mon}}\\ \deg d = j \bmod n}} \frac{H(d,m)}{|m|^{s_1}|d|^{s_2}}. \tag{4.1}$$

We further introduce the notation

$$Z(s_1, s_2; i, *) = \sum_j Z(s_1, s_2; i, j)$$

and

$$Z(s_1, s_2; *, j) = \sum_i Z(s_1, s_2; i, j).$$

These series are absolutely convergent for $\mathrm{Re}(s_1), \mathrm{Re}(s_2) > 3/2$. In fact, we can do a little better. Summing over d first yields

$$\begin{aligned} Z(s_1, s_2; i, *) &= (1-q^{n-ns_1})^{-1}(1-q^{n-ns_2})^{-1}(1-q^{2n-ns_1-ns_2})^{-1} \\ &\quad\times \sum_{\substack{m\in\mathcal{O}_{\mathrm{mon}}\\ \deg m = i \bmod n}} \left(\frac{1}{|m|^{s_1}} \sum_{d\in\mathcal{O}_{\mathrm{mon}}} \frac{H(d,m)}{|d|^{s_2}} \right) \\ &= (1-q^{n-ns_1})^{-1}(1-q^{2n-ns_1-ns_2})^{-1} \sum_{\substack{m\in\mathcal{O}_{\mathrm{mon}}\\ \deg m = i \bmod n}} \frac{E(s_2, m)}{|m|^{s_1}} \end{aligned} \tag{4.2}$$

By the convexity bound of Proposition 3.5, this representation of $Z(s_1, s_2; i, *)$ is seen to meromorphic for $\mathrm{Re}(s_1) > 0, \mathrm{Re}(s_2) > 2$. Alternatively, summing over m first we deduce that $Z(s_1, s_2; i, *)$ is meromorphic for $\mathrm{Re}(s_2) > 0, \mathrm{Re}(s_1) > 2$. Let $\mathcal{R}$ be the tube domain that is the union of these three regions of initial meromorphy:

$$\begin{aligned} \mathcal{R} = \{\mathrm{Re}(s_1), \mathrm{Re}(s_2) > 3/2\} &\cup \{\mathrm{Re}(s_1) > 0, \mathrm{Re}(s_2) > 2\} \\ &\cup \{\mathrm{Re}(s_2) > 0, \mathrm{Re}(s_1) > 2\}. \end{aligned}$$

Let the Weyl group W act on $\mathbb{C}^2$ by

$$\sigma_1 : (s_1, s_2) \mapsto (2-s_1, s_1+s_2-1), \quad \sigma_2 : (s_1, s_2) \mapsto (s_1+s_2-1, 2-s_2). \tag{4.3}$$

Let $\mathcal{F}$ be the real points of a closed fundamental domain for the action of W on $\mathbb{C}^2$:

$$\mathcal{F} = \{\mathrm{Re}(s_1), \mathrm{Re}(s_2) \geq 1\}.$$

One can easily see that $\mathcal{R} \smallsetminus \mathcal{F} \cap \mathcal{R}$ is compact. Therefore, by the principle of analytic continuation and Bochner's tube theorem [1], to prove that $Z(s_1, s_2)$ has a meromorphic continuation to $\mathbb{C}^2$ it suffices to show that the functions $Z(s_1, s_2; i, j)$ satisfy functional equations as (s_1, s_2) goes to $(2-s_1, s_1+s_2-1)$ and $(s_1 + s_2 - 1, 2 - s_2)$. For details, we refer to [2, Section 3].

To prove the σ_2 functional equation, we begin with (4.2) and write

$$\begin{aligned}
Z(s_1, s_2; i, *) &= (1 - q^{n-ns_1})^{-1}(1 - q^{2n-ns_1-ns_2})^{-1} \sum_{\substack{m \in \mathcal{O}_{\mathrm{mon}} \\ \deg m = i \bmod n}} \frac{E(s_2, m)}{|m|^{s_1}} \\
&= (1 - q^{n-ns_1})^{-1}(1 - q^{2n-ns_1-ns_2})^{-1} \\
&\quad \times \sum_{\substack{m \in \mathcal{O}_{\mathrm{mon}} \\ \deg m = i \bmod n}} \frac{|m|^{1-s_2}}{|m|^{s_1}} \sum_{j=0}^{n-1} T_{ji}(s_2) E(2 - s_2, m; j), \quad \text{by Thm. 3.4} \\
&= \sum_{j=0}^{n-1} T_{ji}(s_2) Z(s_1 + s_2 - 1, 2 - s_2; i, j)
\end{aligned}$$

The σ_1 functional equation is proved similarly.

We conclude that

Theorem 4.1. *The double Dirichlet series has a meromorphic continuation to $s_1, s_2 \in \mathbb{C}$ and is holomorphic away from the hyperplanes*

$$s_1 = 1 \pm \tfrac{1}{n}, s_2 = 1 \pm \tfrac{1}{n} \text{ and } s_1 + s_2 = 2 \pm \tfrac{1}{n}.$$

Furthermore, $Z(s_1, s_2)$ satisfies the functional equations

$$\begin{aligned}
Z(s_1, s_2) &= \sum_{i,j} T_{ji}(s_2) Z(s_1 + s_2 - 1, 2 - s_2; i, j) \\
&= \sum_{i,j} T_{ij}(s_1) Z(2 - s_1, s_1 + s_2 - 1; i, j).
\end{aligned}$$

References

1. S. Bochner, *A theorem on analytic continuation of functions in several variables*, Ann. of Math. (2) 39 (1938), no. 1, 14–19.
2. B. Brubaker, D. Bump, G. Chinta, S. Friedberg, J. Hoffstein, *Weyl group multiple Dirichlet series I,* in Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Proc. Sympos. Pure Math., 75, Amer. Math. Soc., Providence, RI, 2006.
3. B. Brubaker, D. Bump, and S. Friedberg, *Weyl group multiple Dirichlet series II: The stable case*, Invent. Math. **165** (2006), 325–355.
4. B. Brubaker, D. Bump, and S. Friedberg. *Weyl group multiple Dirichlet Series, Eisenstein series and crystal bases*, Submitted.

5. B. Brubaker, D. Bump, and S. Friedberg. *Weyl Group Multiple Dirichlet Series: Type A Combinatorial Theory*, Submitted.
6. B. Brubaker, D. Bump, S. Friedberg, and J. Hoffstein, *Weyl group multiple Dirichlet series. III. Eisenstein series and twisted unstable A_r*, Ann. of Math. (2) **166** (2007), no. 1, 293–316.
7. G. Chinta, *Mean values of biquadratic zeta functions*, Invent. Math. **160** (2005), 145–163.
8. G. Chinta. *Multiple Dirichlet series over rational function fields*, Acta Arith., 132(4):377–391, 2008.
9. G. Chinta, S. Friedberg and J. Hoffstein, *Multiple Dirichlet series and automorphic forms,* in Multiple Dirichlet Series, Automorphic Forms, and Analytic Number Theory, Proc. Sympos. Pure Math., 75, Amer. Math. Soc., Providence, RI, 2006.
10. Gautam Chinta, Solomon Friedberg, and Paul E. Gunnells. *On the p-parts of quadratic Weyl group multiple Dirichlet series*, J. Reine Angew. Math., 623:1–23, 2008.
11. G. Chinta and P. E. Gunnells, *Weyl group multiple Dirichlet series constructed from quadratic characters,* Invent. Math. 167 (2007), no.2, 327–353.
12. G. Chinta and P. E. Gunnells, *Constructing Weyl group multiple Dirichlet series,* J. Amer. Math. Soc., to appear.
13. B. Fisher and S. Friedberg, *Double Dirichlet series over function fields*, Compos. Math. **140** (2004), no. 3, 613–630.
14. S. Friedberg, J. Hoffstein, and D. Lieman, *Double Dirichlet series and the n-th order twists of Hecke L-series,* Math. Ann. 327 (2003), no. 2, 315–338.
15. J. Hoffstein, *Theta functions on the n-fold metaplectic cover of* SL(2)*—the function field case*, Invent. Math. 107 (1992), no. 1, 61–86.
16. J. Hoffstein and M. Rosen, *Average values of L-series in function fields*, J. Reine Angew. Math. 426 (1992), 117–150.
17. T. Kubota, *Some results concerning reciprocity law and real analytic automorphic functions,* 1969 Number Theory Institute (Proc. Sympos. Pure Math., Vol. XX, State Univ. New York, Stony Brook, N.Y., 1969), pp. 382–395. Amer. Math. Soc., Providence, R.I., 1971.
18. T. Kubota, *Some number-theoretical results on real analytic automorphic forms,* Several Complex Variables, II (Proc. Internat. Conf., Univ. Maryland, College Park, Md., 1970), pp. 87–96. Lecture Notes in Math., Vol. 185, Springer, Berlin, 1971.
19. R. Livné and S. J. Patterson, *The first moment of cubic exponential sums,* Invent. Math. 148, 79–116 (2002).
20. S. J. Patterson. *Note on a paper of J. Hoffstein* Glasg. Math. J., 49(2):243–255, 2007.

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