ON THE TOPOLOGICAL COMPUTATION OF K_4 OF THE GAUSSIAN AND EISENSTEIN INTEGERS

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Abstract. In this paper we use topological tools to investigate the structure of the algebraic K-groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$, where $i := \sqrt{-1}$ and $\rho := (1 + \sqrt{-3})/2$. We exploit the close connection between homology groups of $\mathrm{GL}_n(R)$ for $n \leqslant 5$ and those of related classifying spaces, then compute the former using Voronoi's reduction theory of positive definite quadratic and Hermitian forms to produce a very large finite cell complex on which $\mathrm{GL}_n(R)$ acts. Our main result is that $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ have no p-torsion for $p \geqslant 5$.

1. Introduction

1.1. **Statement of results.** Let R be the ring of integers of a number field F. Only very few cases are known where the algebraic K-group $K_4(R)$ has been explicitly computed, the first such $K_4(\mathbb{Z})$ having been determined as recently as 2000 by Rognes [17], building on work of Soulé [18]. The goal of this paper is the explicit topological computation of the torsion (away from 2 and 3) in the groups $K_4(R)$ for R one of two special imaginary quadratic examples: the *Gaussian integers* $\mathbb{Z}[i]$ and the *Eisenstein integers* $\mathbb{Z}[\rho]$, where $i := \sqrt{-1}$ and $\rho := (1 + \sqrt{-3})/2$. Our work is in the spirit of Lee–Szczarba [12–14], Soulé [19], and Elbaz-Vincent–Gangl–Soulé [7,8] who treated $K_N(\mathbb{Z})$ for small N, and Staffeldt [20] who investigated $K_3(\mathbb{Z}[i])$. As in these works, the first step is to compute the cohomology of $\mathrm{GL}_n(R)$ for $n \leqslant N+1$; information from this computation is then assembled into information about the K-groups following the program in §1.2. Using these computations we show the following (Theorem 4.1):

Theorem 1.1. *The orders of the groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ are not divisible by any primes $p \geqslant 5$.*

We remark that this result is not new; in fact, Kolster's work [11] implies the stronger result that $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ vanish. Indeed, if R is the ring of integers of a CM field, then Kolster proved that, assuming the Quillen–Lichtenbaum conjecture, the orders of the groups $K_{4n}(R)$, $n = 1, 2, 3, \ldots$, can be computed in terms of special values of certain L-functions. This deep connection between K-groups and special values of L-functions is now a theorem, thanks to the celebrated work by Voevodsky [21] and Rost, as put into context in [9].

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Our work, on the other hand, treats $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ by completely different methods. We only use the definition of the K-groups and explicit results about the cohomology of the relevant arithmetic groups [6], together with Arlettaz's bounds on the kernel of the Hurewicz homomorphism [1], to prove Theorem 4.1. This also explains why our calculations do not allow us to say anything for the primes 2 and 3: both the results of [6] and the injectivity of the Hurewicz map in our cases only hold away from these primes.

1.2. **Outline of method.** In the rest of this introduction we outline the main steps of our argument. These follow the classical approach for computing algebraic K-groups of number rings due to Quillen [15], which shifts the focus to computing the homology (with nontrivial coefficients) of certain arithmetic groups.

(i) **(Definition)** By definition the algebraic K-group $K_N(R)$ of a ring R is a particular homotopy group of a topological space associated to R: we have $K_N(R) = \pi_{N+1}(BQ(R))$, where $BQ(R)$ is a certain classifying space attached to the infinite general linear group $\mathrm{GL}(R)$. In particular $BQ(R)$ is the classifying space of the category $Q(R)$ of finitely generated R-modules. This is known as Quillen's *Q-construction* of algebraic K-theory [16].

(ii) **(Homotopy to Homology)** The Hurewicz homomorphism $\pi_{N+1}(BQ(R)) \to H_{N+1}(BQ(R))$ allows one to replace the homotopy group by a homology group without losing too much information; more precisely, what may get lost is information about small torsion primes appearing in its finite kernel.

(iii) **(Stability)** By a stability result of Quillen [15, p. 198] one can pass from $Q(R)$ to the category $Q_{N+1}(R)$ of finitely generated R-modules of rank $\leqslant N + 1$ for sufficiently large N. This amounts to passing from $\mathrm{GL}(R)$ to the finite-dimensional general linear group $\mathrm{GL}_{N+1}(R)$.

(iv) **(Sandwiching)** The homology groups to be determined are then $H_*(BQ_n(R))$ for $n \leqslant N + 1$. Rather than compute these directly, one uses the fact that they can be sandwiched between homology groups of $\mathrm{GL}_n(R)$, where the homology is taken with (nontrivial) coefficients in the Steinberg module St_n associated to $\mathrm{GL}_n(R)$.

(v) **(Voronoi homology)** The standard method to compute the homology groups $H_m(\mathrm{GL}_n(R), St_n)$ for a number ring R is via *Voronoi complexes*. These are the chain complexes of certain explicit polyhedral reduction domains of a space of positive definite quadratic or Hermitian forms of a given rank, depending respectively on whether $R = \mathbb{Z}$ or R is imaginary quadratic. The Voronoi complex provides most of the desired information on the homology in question: as in (iv), one might again lose information about small primes—in particular, such information could be hidden in the higher differentials of a spectral sequence involving the stabilizers of cells in the Voronoi complex. In any case, one can usually find a small upper bound on the sizes of those primes, which means that one can effectively determine the homology and ultimately the K-groups modulo small primes.

(vi) **(Vanishing Results)** There are various techniques to show vanishing of homology groups. As a starting point one has vanishing results for $H_n(BQ_1)$ as in Theorem 3.1 below, and for $H_0(GL_n, St_n)$ as in Lee–Szczarba [14].

For a given N, using (ii) and knowing the results of (iv)–(vi) for all $0 \leqslant n \leqslant N+1$ is often enough to give a bound $p \leqslant B$ on the primes p dividing the order of the torsion subgroup $K_{N,\mathrm{tors}}(R)$ of $K_N(R)$.

1.3. **Outline of paper.** In this paper the sections work backwards through the method outlined in §1.2 to determine the structure of $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$. In §2, we describe the computation of the Voronoi homology of these two number rings (i.e., step (v) above). In §3 we use the Voronoi homology and some vanishing results to determine the groups $H_m(BQ_n(R))$ (i.e., step (iv) above). A key role here is played by Quillen's stability result (iii) for BQ_n, which serves as a stopping criterion. Finally, in §4 we work out the potential primes entering the kernel of the Hurewicz homomorphism (i.e., step (ii) above), which gives Theorem 1.1.

2. Homology of Voronoi complexes

We first collect the results from [6] concerning the Voronoi complexes attached to $\Gamma = \mathrm{GL}_n(\mathbb{Z}[i])$ or $\Gamma = \mathrm{GL}_n(\mathbb{Z}[\rho])$; this is the necessary information needed for step (v) from §1.2 above. More details about these computations, including background about how the computations are performed, can be found in [6].

Let F be an imaginary quadratic field with ring of integers R, and let $X_n := \mathrm{GL}_n(\mathbb{C})/\mathrm{U}(n)$ be the symmetric space of $\mathrm{GL}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$. The space X_n can be realized as the quotient of the cone of rank n positive definite Hermitian matrices C_n modulo homotheties (i.e. non-zero scalar multiplication), and a partial Satake compactification X_n^* of X_n is given by adjoining boundary components to X_n given by the cones of positive semi-definite Hermitian forms with an F-rational nullspace (again taken up to homotheties). We let $\partial X_n^* := X_n^* \smallsetminus X_n$ denote the *boundary* of X_n^*. Then $\Gamma := \mathrm{GL}_n(R)$ acts by left multiplication on both X_n and X_n^*, and the quotient $\Gamma \backslash X_n^*$ is a compact Hausdorff space.

A generalization—due to Ash [2, Chapter II] and Koecher [10]—of the polyhedral reduction theory of Voronoi [22] yields a Γ-equivariant explicit decomposition of X_n^* into (Voronoi) cells. Moreover, there are only finitely many cells modulo Γ. Let $\Sigma_d^* := \Sigma_d(\Gamma)^*$ be a set of representatives of the Γ-inequivalent d-dimensional Voronoi cells that meet the interior X_n, and let $\Sigma_d := \Sigma_d(\Gamma)$ be the subset of representatives of the Γ-inequivalent *orientable* cells in this dimension; here we call a cell *orientable* if all the elements in its stabilizer group preserve its orientation. One can form a chain complex Vor_*, the *Voronoi complex*, and one can prove that modulo small primes the homology of this complex is the homology $H_*(\Gamma, St_n)$, where St_n is the rank n *Steinberg module* (cf. [4, p. 437]). To keep track of these small primes explicitly, we make the following definition.

Definition 2.1 (Serre class of small prime power groups)**.** *Given $k \in \mathbb{N}$, we let $\mathcal{S}_{p\leqslant k}$ denote the Serre class of finite abelian groups G whose cardinality $|G|$ has all of its prime divisors p satisfying $p \leqslant k$.*

For any finitely generated abelian group G, there is a unique maximal subgroup $G_{p\leqslant k}$ of G in the Serre class $\mathcal{S}_{p\leqslant k}$. We say that two finitely generated abelian groups G and G' are ***equivalent modulo*** $\mathcal{S}_{p\leqslant k}$ *and write $G \simeq_{/p\leqslant k} G'$ if the quotients $G/G_{p\leqslant k} \cong G'/G'_{p\leqslant k}$ are isomorphic.*

Theorem 2.2 ([6, Theorem 3.7])**.** *Let b be an upper bound on the torsion primes for $\mathrm{GL}_n(R)$. Then $H_m(\mathrm{Vor}_*) \simeq_{/p\leqslant b} H_{m-n+1}(\mathrm{GL}_n(R), St_n)$.*

2.1. **Voronoi data for $R = \mathbb{Z}[i]$.** We now give results for the Voronoi complexes and their homology in the cases relevant to our paper. This subsection treats the Gaussian integers; in §2.2 we treat the Eisenstein integers.

Theorem 2.3 ([20]).

1. *There is one d-dimensional Voronoi cell for* $\mathrm{GL}_2(\mathbb{Z}[i])$ *for each* $1 \leqslant d \leqslant 3$, *and only the 3-dimensional cell is orientable.*
2. *The number of d-dimensional Voronoi cells for* $\mathrm{GL}_3(\mathbb{Z}[i])$ *is given by:*

d	2	3	4	5	6	7	8
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[i]))^*\rvert$	2	3	4	5	3	1	1
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[i]))\rvert$	0	0	1	4	3	0	1

Theorem 2.4 ([6, Table 12]). *The number of d-dimensional Voronoi cells for* $\mathrm{GL}_4(\mathbb{Z}[i])$ *is given by:*

d	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[i]))^*\rvert$	4	10	33	98	258	501	704	628	369	130	31	7	2
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[i]))\rvert$	0	0	5	48	189	435	639	597	346	120	22	2	2

We remark that for $\mathrm{GL}_3(\mathbb{Z}[i])$ the Voronoi complexes and their homology ranks were originally computed by Staffeldt [20], who even distilled the 3-part for each homology group. After calculating the differentials for this complex one obtains the following homology groups, in agreement with Staffeldt's results:

Theorem 2.5 ([20, Theorems IV, 1.3 and 1.4, p.785]).

$$H_m(\mathrm{GL}_2(\mathbb{Z}[i]), St_2) \simeq_{/p\leqslant 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise}, \end{cases} \tag{1}$$

$$H_m(\mathrm{GL}_3(\mathbb{Z}[i]), St_3) \simeq_{/p\leqslant 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise}. \end{cases} \tag{2}$$

In particular, from the above theorem we deduce that the only possible torsion primes for $H_m(\mathrm{GL}_n(\mathbb{Z}[i]), St_n)$ for $n = 2, 3$ are the primes 2 and 3.

For $\mathrm{GL}_4(\mathbb{Z}[i])$, the last column of [6, Table 12] shows that the elementary divisors of all the differentials in the Voronoi complex are supported on primes $\leqslant 5$. In fact a closer examination of this table reveals the following:

Theorem 2.6 ([6, Theorem 7.2 and Table 12]).

$$H_m(\mathrm{GL}_4(\mathbb{Z}[i]), St_4) \simeq_{/p\leqslant 5} \begin{cases} \mathbb{Z}^2 & \text{if } m = 5, \\ \mathbb{Z} & \text{if } m = 4, 7, 8, 10, 13, \\ 0 & \text{otherwise}. \end{cases} \tag{3}$$

Moreover, the only degrees where 5*-torsion could occur are* $m = 1, 6$ *or* $m \geqslant 10$.

From this we see that there is the potential for 5-torsion for $H_m(\mathrm{GL}_4(\mathbb{Z}[i]), St_4)$. While there is 5-torsion in H_{10}, and possibly further 5-torsion in H_m for $m \geqslant 6$, we will show that for degree $m = 1$ (the only relevant degree for the K-groups we consider) the group H_1 contains no 5-torsion (Proposition 2.7).

In order to analyze $H_1(\mathrm{GL}_4(\mathbb{Z}[i]), St_4)$ more closely, we will need to use spectral sequences. According to [5, VII.7] there is a spectral sequence $E^r_{d,q}$ converging to the equivariant homology groups $H^\Gamma_{d+q}(X^*_n, \partial X^*_n; \mathbb{Z})$ of the homology pair $(X^*_n, \partial X^*_n)$, and such that

$$E^1_{d,q} = \bigoplus_{\sigma \in \Sigma^*_d} H_q(\Gamma_\sigma, \mathbb{Z}_\sigma),$$

where $\mathbb{Z}_\sigma$ is the orientation module of the cell σ.

Proposition 2.7. *The group* $H_1(\mathrm{GL}_4(\mathbb{Z}[i]), St_4) \simeq_{/p \leqslant 3} \{0\}$.

Proof. Since $H_1(\mathrm{GL}_4(\mathbb{Z}[i]), St_4)$ is a subquotient of $\bigoplus_{d+q=4} E^1_{d,q}$, we consider the individual summands $E^1_{d,4-d}$ for $0 \leqslant p \leqslant 4$:

- Since there are no cells in Σ^*_d for $d \leqslant 2$, we have $E^1_{0,4} = E^1_{1,3} = E^1_{2,2} = 0$.
- Consider now $d = 3$. There are four cells in Σ^*_3, and for each of them the index 2 subgroup acting trivially on the orientation module has an abelianization $\mathbb{Z}/3\mathbb{Z}$ up to 2-groups. Thus in particular we have

$$E^1_{3,1} = \bigoplus_{\sigma \in \Sigma^*_3} H_1(\mathrm{Stab}_\sigma, \mathbb{Z}_\sigma) \in \mathcal{S}_{p \leqslant 3},$$

  where $\mathcal{S}_{p \leqslant 3}$ is as in Definition 2.1, and therefore this term contains no 5-torsion.
- Finally, for $d = 4$, there is only one cell (out of ten) in Σ^*_4, denoted by σ^1_4, that contains a subgroup of order 5. We must therefore show that there is no 5-torsion in $H_1(\mathrm{Stab}(\sigma^1_4), \tilde{\mathbb{Z}})$ (where $\tilde{\mathbb{Z}}$ is the orientation module $\mathbb{Z}_{\sigma^1_4}$). Indeed, the order-preserving subgroup K_1 of $\mathrm{Stab}(\sigma^1_4)$ is isomorphic to $\mathbb{Z}/4\mathbb{Z} \times A_5$, where A_5 is the alternating group on five letters, with abelianization $H_1(\mathrm{Stab}(\sigma^1_4), \tilde{\mathbb{Z}}) = H_1(K_1, \mathbb{Z}) \simeq \mathbb{Z}/4\mathbb{Z}$, which lies in $\mathcal{S}_{p \leqslant 3}$. Thus there can be no 5-torsion from here, which completes the proof.

□

2.2. **Voronoi homology data for** $R = \mathbb{Z}[\rho]$**.** Now we turn to the Eisenstein case.

Theorem 2.8 ([6, Tables 1 and 11])**.**

1. *There is one d-dimensional Voronoi cell for* $\mathrm{GL}_2(\mathbb{Z}[\rho])$ *for each* $1 \leqslant d \leqslant 3$, *and only the 3-dimensional cell is orientable.*
2. *The number of d-dimensional Voronoi cells for* $\mathrm{GL}_3(\mathbb{Z}[\rho])$ *is given by:*

d	2	3	4	5	6	7	8
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[\rho]))^*\rvert$	1	2	3	4	3	2	2
$\lvert\Sigma_d(\mathrm{GL}_3(\mathbb{Z}[\rho]))\rvert$	0	0	1	2	1	1	2

3. *The number of d-dimensional Voronoi cells for* $\mathrm{GL}_4(\mathbb{Z}[\rho])$ *is given by:*

d	3	4	5	6	7	8	9	10	11	12	13	14	15
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[\rho]))^*\rvert$	2	5	12	34	82	166	277	324	259	142	48	15	5
$\lvert\Sigma_d(\mathrm{GL}_4(\mathbb{Z}[\rho]))\rvert$	0	0	0	8	50	129	228	286	237	122	36	10	5

After calculating the differentials we find the same results as for the homology of $\mathbb{Z}[i]$ above:

Theorem 2.9 ([6, Theorems 7.1 and 7.2 with Propositions 3.2 and 3.6]).

$$H_m(\mathrm{GL}_2(\mathbb{Z}[\rho]), St_2) \simeq_{/p\leqslant 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, \\ 0 & \text{otherwise}, \end{cases} \tag{4}$$

$$H_m(\mathrm{GL}_3(\mathbb{Z}[\rho]), St_3) \simeq_{/p\leqslant 3} \begin{cases} \mathbb{Z} & \text{if } m = 2, 3, 6, \\ 0 & \text{otherwise}, \end{cases} \tag{5}$$

$$H_m(\mathrm{GL}_4(\mathbb{Z}[\rho]), St_4) \simeq_{/p\leqslant 5} \begin{cases} \mathbb{Z}^2 & \text{if } m = 5, \\ \mathbb{Z} & \text{if } m = 4, 7, 8, 10, 13, \\ 0 & \text{otherwise}. \end{cases} \tag{6}$$

Proof. Since the ranks of the homology groups in question have been computed in [6], we only have to consider the torsion in the respective groups. For fixed n, any torsion prime of the homology groups $H_m(\mathrm{GL}_n(\mathbb{Z}[\rho]), St_n)$ must either divide the order of the stabilizer of some cell in Σ_d^* for appropriate d, or must divide an elementary divisor of the differentials in the corresponding Voronoi complex. We consider these two possibilities in turn.

First we consider the stabilizers. For ranks $n = 2, 3$, all stabilizers of cells in Σ_d^* lie in $\mathcal{S}_{p\leqslant 3}$. For rank $n = 4$, the prime $p = 5$ is the only torsion prime > 3 occurring for stabilizer orders in Σ_d^*, more precisely it occurs for $d = 9$ (two cells), $d = 14$ (two cells) and $d = 15$ (one cell).

Next we consider elementary divisors. In rank $n = 2$, the elementary divisors occurring are all even, and apart from $m = 2$, where $H_2(\mathrm{GL}_2(\mathbb{Z}[\rho]), St_2) = H_3(\mathrm{Vor}_*) = \mathbb{Z}$ modulo $\mathcal{S}_{p\leqslant 3}$, we have $H_m(\mathrm{GL}_2(\mathbb{Z}[\rho]), St_2) = H_{m+1}(\mathrm{Vor}_*) = 0$ modulo $\mathcal{S}_{p\leqslant 3}$. In rank $n = 3$, the only non-trivial elementary divisor for any of the differentials involved is 9, arising from $d_8^1 : E_{8,0}^1 \longrightarrow E_{7,0}^1$. Moreover, we get $H_m(\mathrm{GL}_3(\mathbb{Z}[\rho]), St_3) = H_{m+2}(\mathrm{Vor}_*) = \mathbb{Z}$ modulo $\mathcal{S}_{p\leqslant 3}$ for $m = 2, 3$ or 6, and is zero otherwise. Finally, for rank $n = 4$, the only torsion prime > 3 for the homology groups $H_{m+3}(\mathrm{Vor}_*)$ is $d = 5$, which divides the elementary divisor 15 of d_{14}^1. This completes the proof. ◻

As with $\mathbb{Z}[i]$, a more refined analysis of the $GL_4(\mathbb{Z}[\rho])$ case shows that H_1 contains no 5-torsion:

Proposition 2.10.

$$H_1(\mathrm{GL}_4(\mathbb{Z}[\rho]), St_4) \simeq_{/p\leqslant 3} \{0\}. \tag{7}$$

Proof. The argument is very similar to that of the proof of Proposition 2.7. In rank 4, we have that $H_1(\mathrm{GL}_4(\mathbb{Z}[\rho]), St_4)$ is a subquotient of

$$\bigoplus_{d+q=4} E_{d,q}^1 = \bigoplus_{d+q=4} \bigoplus_{\sigma\in\Sigma_d^*} H_q(\Gamma_\sigma, \mathbb{Z}_\sigma). \tag{8}$$

We consider each of these summands in turn.

If $d \leqslant 2$, then there are no cells of dimension d to worry about. For $d = 3$, there are two cells in Σ_3^*, with stabilizer in $\mathcal{S}_{p\leqslant 3}$, and hence

$$E_{3,1}^1 = \bigoplus_{\sigma\in\Sigma_3^*} H_1(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma) \in \mathcal{S}_{p\leqslant 3}.$$

Finally suppose $d = 4$. Then $E^1_{4,0} = 0 \mod \mathcal{S}_{p\leqslant 2}$, as none of the 5 classes in dim 4 is orientable. Thus modulo $\mathcal{S}_{p\leqslant 3}$ all summands in (8) vanish, which completes the proof. ◻

3. Vanishing and sandwiching

In this section, we carry out the sandwiching argument (step (iv) of §1.2). As a first step we invoke a vanishing result for homology groups for BQ_1 due to Quillen [15, p.212]. In our cases this result boils down to the following statement:

Theorem 3.1. *For the rings $R = \mathbb{Z}[i]$ and $\mathbb{Z}[\rho]$, we have*

$$H_n(BQ_1) = 0 \qquad \textit{whenever } n \geqslant 3\,.$$

For $R = \mathbb{Z}[i]$ a slightly stronger result is proved in [20, Lemma I.1.2]. However, we will not need this stronger result for $\mathbb{Z}[i]$, or its analogue for $\mathbb{Z}[\rho]$.

Using our homology data from §2 and Theorem 3.1, we can get for both rings $R = \mathbb{Z}[i]$ and $R = \mathbb{Z}[\rho]$ the following result:

Proposition 3.2. *$H_5(BQ) \simeq_{/p\leqslant 3} \mathbb{Z}$.*

Proof. We will successively determine $H_5(BQ_j)$ for $j = 1, \dots, 5$ and then identify the last group via stability with $H_5(BQ)$. For this, we will combine results from §2 with Quillen's long exact sequence for different r, given by

(9)
$$\cdots \longrightarrow H_n(BQ_{r-1}) \longrightarrow H_n(BQ_r) \longrightarrow H_{n-r}(\mathrm{GL}_r, St_r) \longrightarrow H_{n-1}(BQ_{r-1}) \longrightarrow \cdots\,.$$

The case $j = 1$. By Theorem 3.1 we have $H_n(BQ_1) = 0$ for $n \geqslant 3$.

The case $j = 2$. From the above sequence (9) for $r = 2$, we get

$$\underbrace{H_5(BQ_1)}_{=0} \longrightarrow H_5(BQ_2) \longrightarrow H_3(\mathrm{GL}_2, St_2) \longrightarrow \underbrace{H_4(BQ_1)}_{=0},$$

whence $H_5(BQ_2) = 0 \mod \mathcal{S}_{p\leqslant 3}$ by (1) and (4).

The case $j = 3$. Now we invoke another result of Staffeldt's who showed (see [20, proof of Theorem I.1.1] that

$$H_4(BQ_2) = H_4(BQ_3) = \mathbb{Z} \mod \mathcal{S}_{p\leqslant 3}\,. \tag{10}$$

From (9) for $r = 3$ we get the exact sequence, working mod $\mathcal{S}_{p\leqslant 3}$,

$$H_5(BQ_2) \longrightarrow H_5(BQ_3) \longrightarrow \underbrace{H_2(\mathrm{GL}_3, St_3)}_{=\mathbb{Z}\ \text{(by (2), (5))}} \longrightarrow \underbrace{H_4(BQ_2)}_{=\mathbb{Z}\ \text{(by (10))}} \longrightarrow \underbrace{H_4(BQ_3)}_{=\mathbb{Z}\ \text{(by (10))}} \longrightarrow \underbrace{H_1(\mathrm{GL}_3, St_3)}_{=0\ \text{(by (2), (5))}}\,.$$

Since the leftmost group $H_5(BQ_2)$ vanishes modulo $\mathcal{S}_{p\leqslant 3}$ by the case $j = 2$, this sequence implies that $H_5(BQ_3) = \mathbb{Z} \mod \mathcal{S}_{p\leqslant 3}$.

The case $j = 4$. Moreover, since $H_2(\mathrm{GL}_4, St_4) = H_1(\mathrm{GL}_4, St_4) = 0 \mod \mathcal{S}_{p\leqslant 3}$ by Theorem 2.6 and Proposition 2.7, the sequence (9) for $r = 4$ gives in a similar way that

$$H_5(BQ_4) = H_5(BQ_3) = \mathbb{Z} \mod \mathcal{S}_{p\leqslant 3}\,. \tag{11}$$

The case $j = 5$. This is the most complicated of all the cases to handle. Note that BQ is an H-space which implies that $H_*(BQ) \otimes \mathbb{Q}$ is the enveloping algebra of $\pi_*(BQ) \otimes \mathbb{Q}$. We know that $K_0(\mathbb{Z}[i]) = \mathbb{Z}$, $K_1(\mathbb{Z}[i]) = \mathbb{Z}/2$ and $K_2(\mathbb{Z}[i]) = 0$

[3, Appendix] as well as $K_3(\mathbb{Z}[i]) = \mathbb{Z} \oplus \mathbb{Z}/24$ (cf. Weibel [23], Theorem 73 in combination with Example 28), so modulo $\mathcal{S}_{p\leqslant 3}$ we have that

$$\pi_1(BQ) \otimes \mathbb{Q} = K_0(\mathbb{Z}[i]) \otimes \mathbb{Q} = \mathbb{Q}\,,$$

that $\pi_2(BQ) \otimes \mathbb{Q} = \pi_3(BQ) \otimes \mathbb{Q} = 0$, and that

$$\pi_4(BQ) \otimes \mathbb{Q} = K_3(\mathbb{Z}[i]) \otimes \mathbb{Q} = \mathbb{Q}\,.$$

Hence $H_5(BQ) \otimes \mathbb{Q}$ contains the product of $\pi_1(BQ) \otimes \mathbb{Q}$ by $\pi_4(BQ) \otimes \mathbb{Q}$ and so its dimension is at least 1.

The stability result foreshadowed in step (iii) of §1.2 (resulting for a Euclidean domain Λ from $H_0(\mathrm{GL}_n(\Lambda), St_n) = 0$ for $n \geqslant 3$, [13, Corollary to Theorem 4.1]), now implies that one has $H_5(BQ) = H_5(BQ_5)$. By the above we get that the rank of $H_5(BQ_5) = H_5(BQ)$ is at least 1.

Therefore, invoking yet again Quillen's exact sequence (9), this time for $r = 5$, and using the above result that $H_5(BQ_4)$ is equal to $\mathbb{Z}$ modulo $\mathcal{S}_{p\leqslant 3}$, we deduce from

$$\underbrace{H_5(BQ_4)}_{=\mathbb{Z} \text{ by } (11)} \longrightarrow H_5(BQ_5) \longrightarrow \underbrace{H_0(\mathrm{GL}_5, St_5)}_{=0}$$

that $H_5(BQ) = H_5(BQ_5)$ must be equal to $\mathbb{Z}$ modulo $\mathcal{S}_{p\leqslant 3}$ as well. Thus $H_5(BQ)$ cannot contain any p-torsion with $p > 3$. $\square$

4. Relating $K_4(\mathcal{O})$ and $H_5(BQ(\mathcal{O}))$ via the Hurewicz homomorphism

It is well known that for a number ring R the space $BQ(R)$ is an infinite loop space. Hence a theorem due to Arlettaz [1, Theorem 1.5] shows that the kernel of the corresponding Hurewicz homomorphism $K_4(R) = \pi_5(BQ) \to H_5(BQ)$ is certainly annihilated by 144 (cf. Definition 1.3 in loc.cit., where this number is denoted R_5). Thus that kernel lies in $\mathcal{S}_{p\leqslant 3}$ (Definition 2.1).

Therefore this Hurewicz homomorphism is injective modulo $\mathcal{S}_{p\leqslant 3}$. For $R = \mathbb{Z}[i]$ or $\mathbb{Z}[\rho]$, Proposition 3.2 implies that $H_5(BQ)$ contains no p-torsion for $p > 3$. After invoking Quillen's result that $K_{2n}(R)$ is finitely generated and Borel's result that the rank of $K_{2n}(R)$ is zero for any number ring R and $n > 0$, we obtain the following theorem:

Theorem 4.1. *The groups $K_4(\mathbb{Z}[i])$ and $K_4(\mathbb{Z}[\rho])$ lie in $\mathcal{S}_{p\leqslant 3}$.*

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