CELLS IN COXETER GROUPS

PAUL E. GUNNELLS

Introduction

Cells—left, right, and two-sided—were introduced by D. Kazhdan and G. Lusztig in their study of the representation theory of Coxeter groups and Hecke algebras [22]. Cells are related to many disparate and deep topics in mathematics, including singularities of Schubert varieties [23], representations of p-adic groups [24], characters of finite groups of Lie type [25], the geometry of unipotent conjugacy classes in simple complex algebraic groups [5,6], composition factors of Verma modules for semisimple Lie algebras [21], representations of Lie algebras in characteristic p [19], and primitive ideals in universal enveloping algebras [32].[1]

In this note we hope to present a different and often overlooked aspect of the cells: as geometric objects in their own right, they possess an evocative and complex beauty. We also want to draw attention to connections between cells and some ideas from theoretical computer science.

Cells are subsets of Coxeter groups, and as such can be visualized using standard tools from the theory of the latter. How this is done, along with some background, is described in the next section. In the meantime we want to present a few examples, so that the reader can quickly see how intriguing cells are.

Let $p, q, r \in \mathbb{N} \cup \{\infty\}$ satisfy $1/p + 1/q + 1/r \leq 1$, where we put $1/\infty = 0$. Let $\Delta = \Delta_{pqr}$ be a triangle with angles $(\pi/p, \pi/q, \pi/r)$. If $1/p + 1/q + 1/r = 1$, then Δ is Euclidean, and can be drawn in $\mathbb{R}^2$; otherwise Δ lives in the hyperbolic plane. In either case, the edges of Δ can be extended to lines, and reflections in these lines are isometries of the underlying plane. The subgroup $W = W_{pqr}$ of the group of isometries generated by these reflections is an example of a Coxeter group. Under the action of W, the images of Δ become a tessellation of the plane, with tiles in bijection with W (Figure 1). Hence we can picture cells by coloring the tiles of this tessellation.

We thank M. Belolipetsky, W. Casselman, J. Humphreys, and E. Sommers for helpful conversations. Some computations to generate the figures were done using software by W. Casselman, F. du Cloux, and D. Holt. In particular, the basic Postscript code to draw polygons in the Poincaré disk is due to W. Casselman, as is the photo of G. Lusztig. The author is partially supported by the NSF.

[1] D. Vogan [32] also introduced cells for Weyl groups like those of Kazhdan-Lusztig.

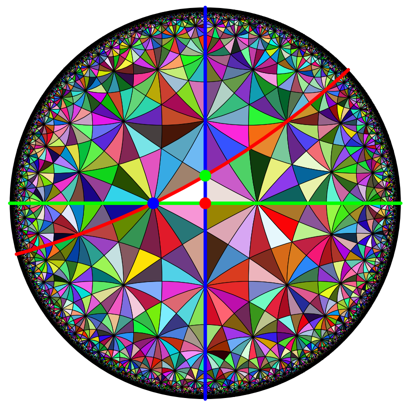


Figure 1. Generating a tessellation of the hyperbolic plane by reflections. The central white tile is repeatedly reflected in the red, green, and blue lines.

For example, the triangle Δ_{236} is Euclidean, and the associated group W_{236} is also known as the affine Weyl group $\widetilde{G}_2$. Figure 2 shows George Lusztig sporting a limited edition T-shirt emblazoned with the two-sided cells of $\widetilde{G}_2$ [26], also reproduced in Figure 3. Figure 4 shows two hyperbolic examples, the groups W_{237} and $W_{23\infty}$. The latter group is also known as the modular group $\mathrm{PSL}_2(\mathbb{Z})$. We invite the reader to ponder how the three pictures are geometrically part of the same family.

Visualizing Coxeter groups

By definition, a Coxeter group[2] W is a group generated by a finite subset $S \subset W$ where the defining relations have the form $(st)^{m_{s,t}} = 1$ for pairs of generators $s, t \in S$. The exponents $m_{s,t}$ are taken from $\mathbb{N} \cup \{\infty\}$, and we require $m_{s,s} = 1$. Hence each generator s is an involution. Two generators s, t commute if and only if $m_{s,t} = 2$.

The most familiar example of a Coxeter group is the *symmetric group* $\mathfrak{S}_n$; this is the group of all permutations of an n-element set $\{1, \ldots, n\}$. We can take S to be the set of simple transpositions s_i, where s_i is the permutation that interchanges i and $i+1$ and fixes the rest. It's not hard to see that S generates $\mathfrak{S}_n$, and that the generators satisfy $(s_i s_{i+1})^3 = 1$ and commute otherwise.

The triangle groups W_{pqr} from the introduction are also Coxeter groups, for which the generators are reflections through the lines spanned by the edges of the fixed triangle Δ_{pqr}. The most important Coxeter groups are certainly the *Weyl* and *affine Weyl* groups, which play a vital role in geometry and algebra. In fact, the symmetric group $\mathfrak{S}_n$ is also known to cognoscenti as the Weyl group A_{n-1}, while the three Euclidean triangle groups W_{333}, W_{234}, and W_{236} are examples of affine Weyl groups.

[2] For more about Coxeter groups, we recommend [8, 20].

FIGURE 2. George Lusztig delivering an Aisenstadt lecture at the CRM during the workshop *Computational Lie Theory* in Spring 2002. This year marks Lusztig's 60th birthday; a conference in his honor will be held at MIT from May 30 to June 3, 2006.

The first step towards a geometric picture of a Coxeter group is its *standard geometric realization*. This is a way to exhibit W as a subgroup of $\mathrm{GL}(V)$, where V is a real vector space of dimension $|S|$. Suppose we have a basis $\Delta = \{\alpha_s \mid s \in S\}$ of the dual space V^*. For each $t \in S$, there is a unique point $\alpha_s^\vee \in V$ such that $\langle \alpha_s, \alpha_t^\vee \rangle = -2\cos(\pi/m_{s,t})$ for all $s \in S$, where the brackets denote the canonical pairing between V^* and V. Each α_s determines a hyperplane H_s, namely the subspace of V on which α_s vanishes. For each s, let $\sigma_s \in \mathrm{GL}(V)$ be the linear map $\sigma_s(v) = v - \langle \alpha_s, v\rangle \alpha_s^\vee$. Note that σ_s fixes H_s and takes $\alpha_s^\vee$ to $-\alpha_s^\vee$ (Figure 5(a)). One can show that the maps $\{\sigma_s \mid s \in S\}$ satisfy $(\sigma_s\sigma_t)^{m_{s,t}} = \mathrm{Id}$, which implies that the map $s \mapsto \sigma_s$ extends to a representation of W. It is known that this representation is *faithful*, and thus we can identify W with its image in $\mathrm{GL}(V)$. [3]

[3]This construction allows us to define Weyl and affine Weyl groups. A Weyl group W is a finite Coxeter group generated by a set S of real reflections and also preserving a certain Euclidean lattice

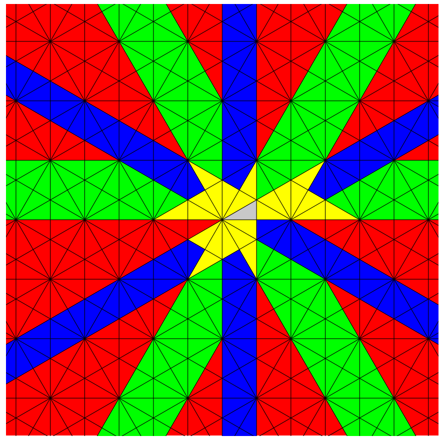
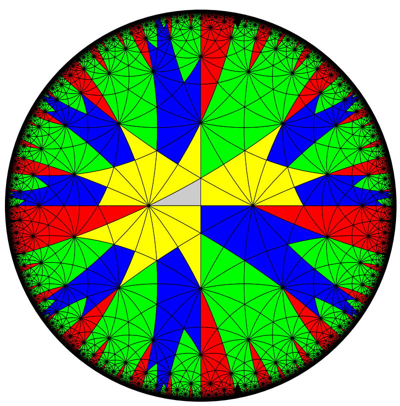
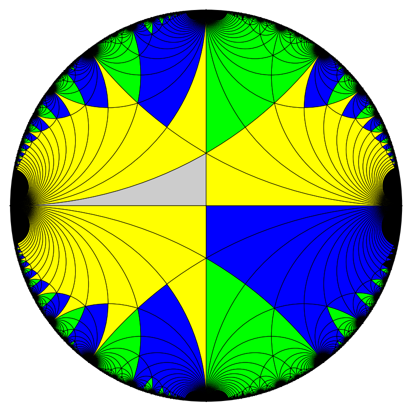


FIGURE 3. $\widetilde{G}_2 = W_{236}$



(a) W_{237}



(b) $W_{23\infty}$

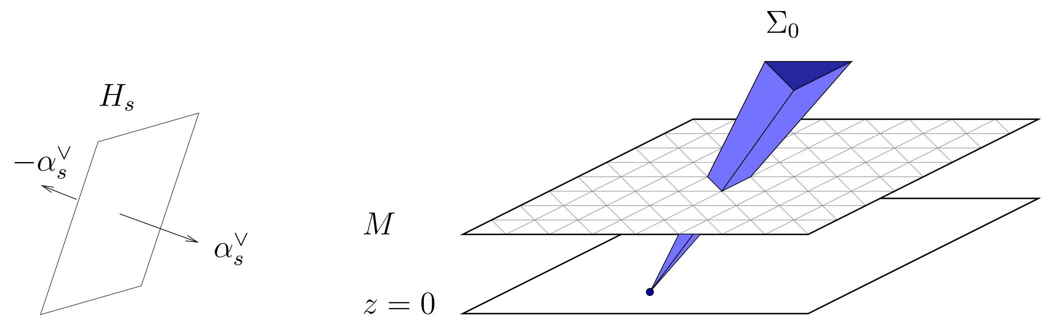
FIGURE 4.

Next we need the *Tits cone* $\mathscr{C} \subset V$. Each hyperplane H_s divides V into two halfspaces. We let H_s^+ be the closed halfspace on which α_s is nonnegative. The intersection $\Sigma_0 = \cap H_s^+$, where s ranges over S, is a closed simplicial cone in V. The closure of the union of all W-translates of Σ_0 is a cone $\mathscr{C}$ in V; this is the Tits cone. It is known that $\mathscr{C} = V$ exactly when W is finite. Usually in fact $\mathscr{C}$ is much less than all of V. Hence the Tits cone gives a better picture for the action of W on V.

L in its geometric realization. The associated affine Weyl group $\widetilde{W}$ is the extension of W by L. As a Coxeter group $\widetilde{W}$ is generated by S and one additional affine reflection.

Under certain circumstances we can obtain a more succinct picture of the action of W on $\mathscr{C}$. For certain groups W it is possible to take a nice "cross-section" of the simplicial cones tiling $\mathscr{C}$ to obtain a manifold M tessellated by simplices. An example can be seen in Figure 5(b) for the affine Weyl group $\widetilde{A}_2$. This group has three generators r, s, t, with the product of any two distinct generators having order three. Thus $V = \mathbb{R}^3$, and the Tits cone $\mathscr{C}$ is the upper halfspace $\{(x, y, z) \in \mathbb{R}^3 \mid z \geq 0\}$. It turns out that the action of $\widetilde{A}_2$ preserves the affine hyperplane $M := \{z = 1\}$, and moreover the intersections of M with translates of Σ_0 are equilateral triangles. This reveals that $\widetilde{A}_2$ is none other than our triangle group W_{333}. A similar picture works for any affine Weyl group, except that the triangles must be replaced by higher-dimensional simplices whose dihedral angles are determined by the exponents $m_{s,t}$.

For more examples we can consider the hyperbolic triangle groups W_{pqr}, where $1/p + 1/q + 1/r < 1$. In this case the Tits cone is a certain round cone in $\mathbb{R}^3$, and the manifold M is one sheet of a hyperboloid (Figure 6). Then M can be identified with the hyperbolic plane; under this identification the intersections $M \cap w\Sigma_0$ become the triangles of our tessellation.



(a) σ_s negates $\alpha_s^\vee$ and fixes H_s

(b) Slicing the Tits cone for $\widetilde{A}_2$

FIGURE 5.

W-GRAPHS AND CELLS

There are two main ingredients needed to define cells: descent sets and Kazhdan-Lusztig polynomials. To introduce them we require a bit more notation.

The Coxeter group (W, S) comes equipped with a *length function* $\ell\colon W \to \mathbb{N} \cup \{0\}$, and a partial order $\leq$, the *Chevalley-Bruhat order*. Any $w \in W$ can be written as a finite product $s_1 \cdots s_N$ of the generators $s \in S$. Such an expression is called *reduced* if we cannot use the relations to produce a shorter expression for w. Then the length $\ell(w)$ is the length N of a reduced expression $s_1 \cdots s_N = w$. The partial

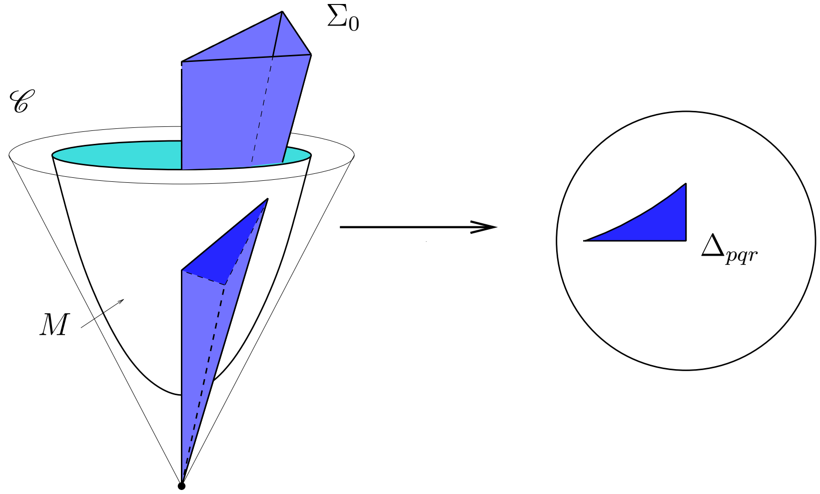


FIGURE 6. Slicing the Tits cone for a hyperbolic triangle group

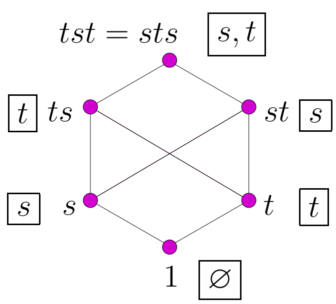
order $\leq$ can also be characterized via reduced expressions. Given an expression $s_1 \cdots s_N$, a *subexpression* is a (possibly empty) expression of the form $s_{i_1} \cdots s_{i_M}$, where $1 \leq i_1 < \cdots < i_M \leq N$. Then $y \leq w$ if an expression for y appears as a subexpression of a reduced expression for w. Although it is not obvious from this definition, this partial order is well-defined.

The *left descent set* $\mathscr{L}(w) \subset S$ of $w \in W$ is simply the set of all generators s such that $\ell(sw) < \ell(w)$. There is an analogous definition for *right descent set*. The definition of the *Kazhdan-Lusztig polynomials*, on the other hand, is too lengthy to reproduce here, although it can be phrased in completely elementary terms. For each pair $y, w \in W$ satisfying $y \leq w$, there is a Kazhdan-Lusztig polynomial $P_{y,w} \in \mathbb{Z}[t]$. By definition $P_{y,y} = 1$; otherwise $P_{y,w}$ has degree at most $d(y,w) := (\ell(w) - \ell(y) - 1)/2$. These subtle polynomials are seemingly ubiquitous in representation theory; they encode deep information about various algebraic structures attached to (W,S). Moreover, computing these polynomials in practice is daunting: memory is rapidly consumed in even the simplest examples. In any case, for our purposes we only need to know whether or not $P_{y,w}$ actually attains the maximum possible degree $d(y,w)$ for a given pair $y < w$. We write y—w if this is so; when $w < y$ we write y—w if w—y holds.

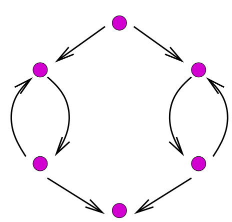
We are finally ready to define cells. The *left W-graph* $\Gamma_{\mathscr{L}}$ of W is the directed graph with vertex set W, and with an arrow from y to w if and only if y—w and $\mathscr{L}(y) \not\subset \mathscr{L}(w)$. The left cells are extracted from the left W-graph as follows. Given any directed graph, we say two vertices are in the same *strong connected component* if there exist directed paths from each vertex to the other. Then the *left cells* of W are exactly the strong connected components of the graph $\Gamma_{\mathscr{L}}$. The *right cells* are defined using the analogously constructed *right W-graph* $\Gamma_{\mathscr{R}}$, while y, w are in the same *two-sided cell* if they are in the same left *or* right cell.

Figure 7 illustrates all the computations necessary to produce the cells for the symmetric group $\mathfrak{S}_3 = \langle s, t \mid s^2 = t^2 = (st)^3 = 1\rangle$. Figure 7(a) shows $\mathfrak{S}_3$ with its partial order and with the left descent sets in boxes. For this group one can compute that $P_{y,w} = 1$ for all relevant pairs (y, w). Thus all the information needed to produce $\Gamma_{\mathscr{L}}$ is contained in the left descent sets. Figure 7(b) shows the resulting graph $\Gamma_{\mathscr{L}}$, and Figure 7(c) shows the four left cells. Computing right descent sets shows that there are three two-sided cells, with the blue and green cells forming a single two-sided cell.

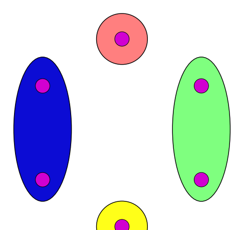
Now we can explain the coloring scheme used in Figures 3 and 4. All regions of a given color comprise a two-sided cell. Moreover, the left cells are exactly the *connected components* of the two-sided cells, in the following sense. Let us say two triangles are *adjacent* if they meet in an edge. Then by definition, a set $\mathscr{T}$ of triangles is connected if for any two triangles $\Delta, \Delta' \in \mathscr{T}$ it is possible within $\mathscr{T}$ to build a sequence $\Delta_* = \Delta_1, \Delta_2, \ldots$ of triangles with each Δ_i adjacent to Δ_{i+1}, and such that the sequence Δ_* contains Δ and Δ'. Note a significant difference between the Euclidean group W_{236} and the two hyperbolic groups. For the former, each two-sided cell contains only finitely many left cells, whereas this is not necessarily the case in general. The latter phenomenon was first observed by R. Bédard [2], who also showed [3] that there are infinitely many left cells for all rank 3 crystallographic hyperbolic Coxeter groups (see the last section for the definition of crystallographic). M. Belolipetsky proved that each Coxeter group in a certain infinite family has infinitely many left cells [4].



(a) $\mathfrak{S}_3$



(b) $\Gamma_{\mathscr{L}}$



(c) Left cells

FIGURE 7.

MORE EXAMPLES

There are two families of Coxeter groups for which we have a good combinatorial understanding of their cells: the symmetric groups $\mathfrak{S}_n$ and the affine Weyl groups $\widetilde{A}_n$.

For the former, left cells appear naturally in the combinatorics literature in the study of the *Robinson-Schensted correspondence*. A lucid exposition of this connection can be found in Chapter 6 of the recently published [8].

The latter is the work of J.-Y. Shi [29]. To describe some of his results, recall that we can associate to the group $\widetilde{A}_n$ a tiling of $\mathbb{R}^n$ by simplices. The simplices can be further grouped into certain convex sets called *sign-type regions*. Figure 8(a) shows the sixteen sign-type regions for $\widetilde{A}_2$; in general for $\widetilde{A}_n$ there are n^{n+2} sign-type regions. One of Shi's main results is that each left cell is a union of sign-type regions. Moreover, Shi also gave an explicit algorithm that allows one to determine to which left cell a given region belongs. The algorithm requires too much notation to state here, but it is completely elementary and involves no computation of Kazhdan-Lusztig polynomials. Figure 8(b) shows the two-sided cells for $\widetilde{A}_2$ [26]; one can clearly see how the regions are joined into cells.

Figures 9(a) and 9(b) depict the cells of $\widetilde{A}_3$.[4] These images were computed directly from the data in [29, §7.3].[5] In the exploded view we have omitted the red cells, which are all simplical cones. Figure 10 shows the left cells up to congruence. All left cells in a given two-sided cell are congruent, except for the yellow two-sided cell, which contains two distinct types of left cells up to congruence (an S and a U).

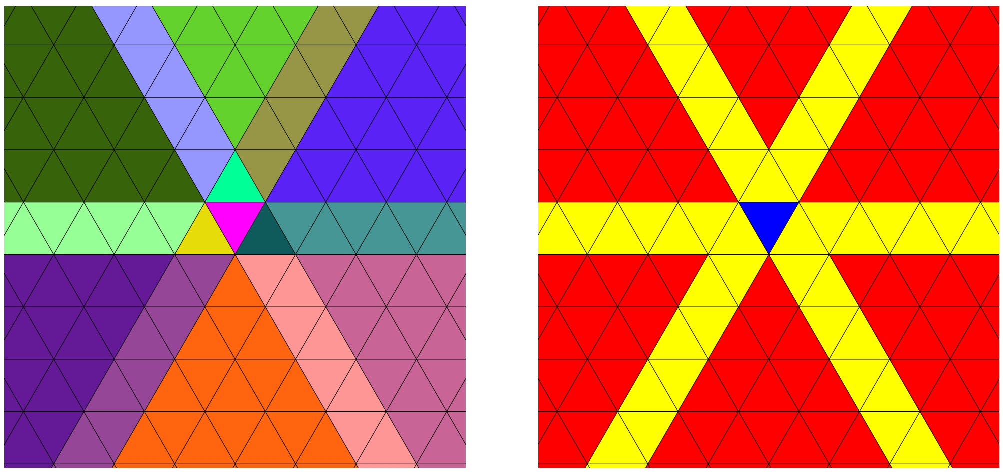
These figures also indicate relationships between cells in different rank groups. Perhaps the most colorful way to describe them is through the *permutahedron*, which is a polytope Π_W attached to a Weyl group W as follows. Let $x \in V$ be a point in the standard geometric realization of W such that the W-orbit of x has size $|W|$. Then Π_W is defined to be the closed convex hull of the points $\{w \cdot x \mid w \in W\}$. It turns out that the combinatorial type of Π_W is independent of the choice of x, and moreover the structure of Π_W is easy to understand: its faces are isomorphic to lower-rank permutahedra $\Pi_{W'}$, where $W' \subset W$ is the subgroup generated by any subset $S' \subset S$ (such subgroups are called *standard parabolic subgroups*). For example, the polytope underlying Figure 9(a) is the permutahedron for the symmetric group $\mathfrak{S}_4$. The eight hexagonal (respectively, six square) faces correspond to parabolic subgroups isomorphic to $\mathfrak{S}_3$ (resp., $\mathfrak{S}_2 \times \mathfrak{S}_2$).

Now the relationship between cells of affine groups of different ranks is conjectured to be as follows. For any finite Weyl group W, let $\widetilde{W}$ be the associated affine Weyl group. Then the intersection of the cells of $\widetilde{W}$ with the face of Π_W corresponding to the standard parabolic subgroup P should produce the picture for the cells of the affine group $\widetilde{P}$. This is clearly visible in Figure 9(a): the cells for $\widetilde{A}_2$ (respectively, $\widetilde{A}_1 \times \widetilde{A}_1$) appear when one slices the cells for $\widetilde{A}_3$ with hexagonal (resp. square) faces of Π_{A_3}. Comparing the cells for $\widetilde{C}_3$ (Figure 11(b)), originally computed by R.

[4] To keep the pictures uncluttered, we have omitted the edges of the simplices.

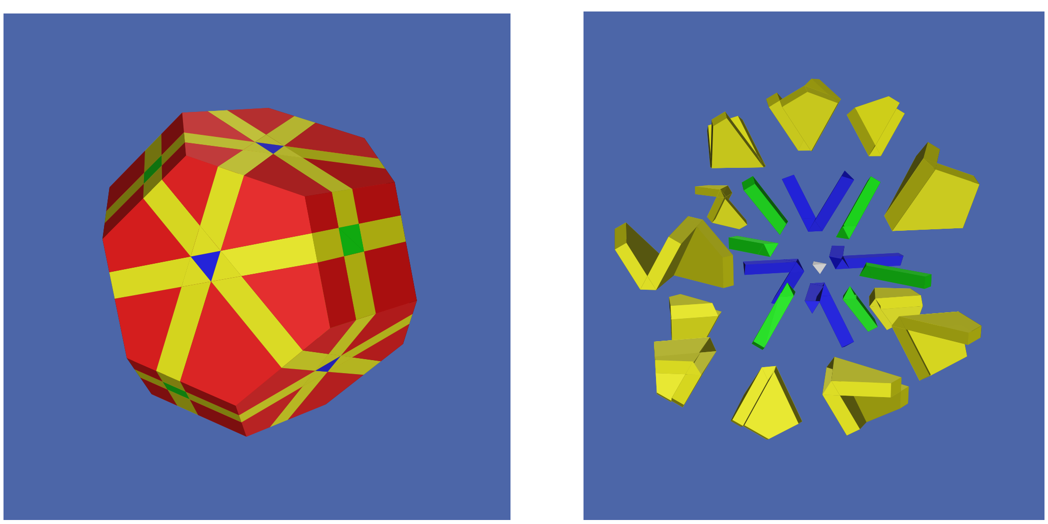
[5] Unfortunately this data is incomplete due to a publisher error: four left cells are missing.

Bédard [2], with the cells of $\widetilde{C}_2$ (Figure 11(a), [26]) shows another example of this. For more along these lines see [17].



(a) $\widetilde{A}_2$ sign-type regions (b) $\widetilde{A}_2$ cells

FIGURE 8



(a) (b)

FIGURE 9. Two views of the cells of $\widetilde{A}_3$

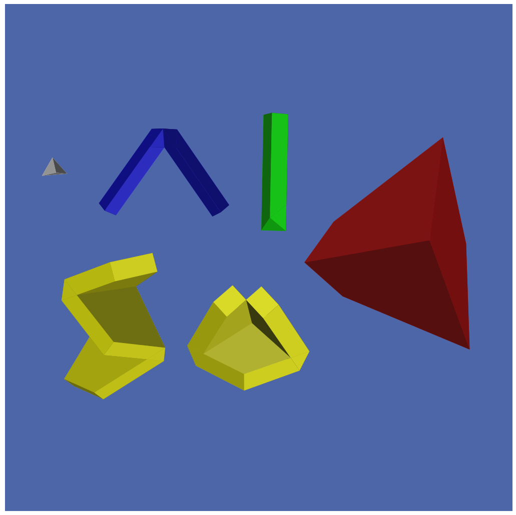
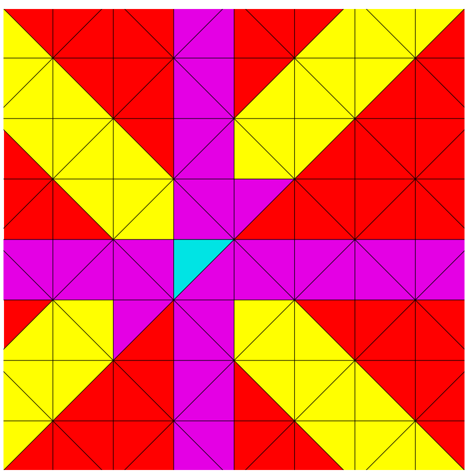
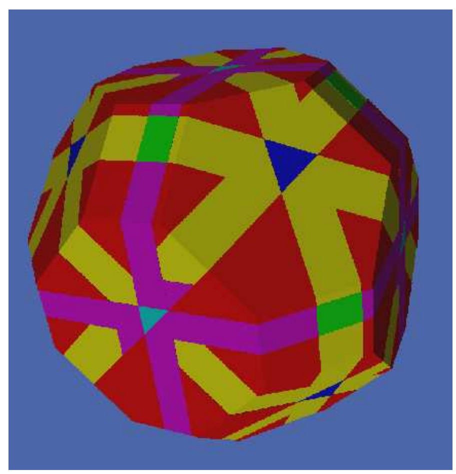


FIGURE 10. $\widetilde{A}_3$ left cells up to congruence



(a) $\widetilde{C}_2$



(b) $\widetilde{C}_3$

FIGURE 11

CELLS AND AUTOMATA

Simple examples show that W-graphs can be quite complicated. However, despite this complexity lurking in their construction, the cells themselves appear to be very regular. In fact, for many groups one can prove that the cells can be built using a relatively small set of rules, rules that involve no Kazhdan-Lusztig polynomial computations at all [13, 14].

Computer scientists have a formal way to work with this phenomenon, the theory of *regular languages* and *finite state automata* [1]. One starts with a finite set A, called an *alphabet*. *Words* over the alphabet are sequences of elements of A, and any set L of words over A is called a *language*. Informally, a language is *regular* if its words can be recognized using a finite list of finite patterns in the alphabet, patterns that are familiar to anyone who has ever used a Unix shell (e.g., `ls *.tex`). A *finite state automaton* F over A is a finite directed graph with edges labelled by elements of A. The vertices of F are called *states*. All vertices are designated as either *accepting* or *nonaccepting*, and one vertex is set to be the initial state.

Such an automaton determines a language over A as follows. One starts at the initial state and follows a directed path terminating at an accepting state. Such a path determines a word (one simply concatenates the labels of the edges along the path to produce a word). We say that this word is *recognized* by the automaton. The set of all words recognized by an automaton is hence a language over A. A basic theorem is that a language is regular exactly when it can be recognized by a finite state automaton.

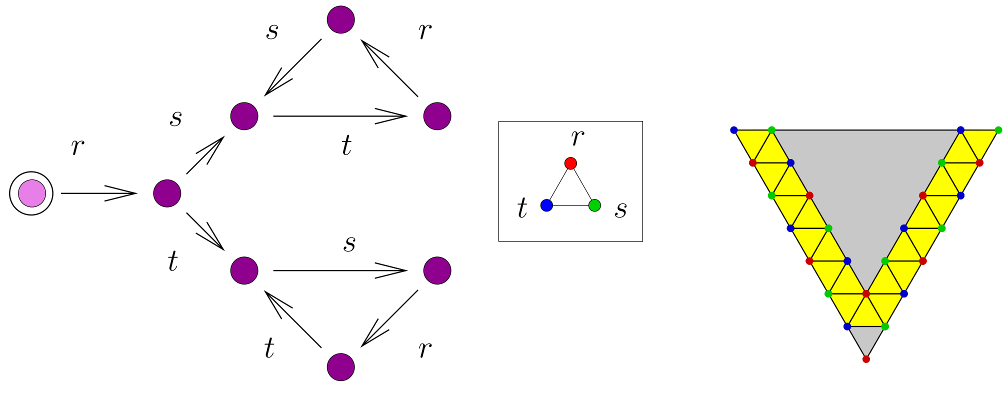
For a Coxeter group W, the alphabet is the set of generators S, and the language is the set $\mathtt{Reduced}_W$ of all reduced expressions. By a result of B. Brink and R. Howlett [10], the language $\mathtt{Reduced}_W$ is regular. Any left cell C determines a sublanguage $\mathtt{Reduced}_W(C) := \{w \in \mathtt{Reduced}_W \mid w \text{ is a word in } C\}$. W. Casselman has conjectured that the language $\mathtt{Reduced}_W(C)$ is always regular.

Figure 12 illustrates these ideas for one of the yellow left cells in $\widetilde{A}_2$ (Figure 8(b)). This cell has the property that every element in it has a *unique* reduced expression; such cells were first considered by G. Lusztig [24, Proposition 3.8]. The automaton has edges labelled by elements of $\{r, s, t\}$. The initial state is the encircled light purple vertex and is nonaccepting; all other vertices are accepting. To make the connection between the automaton and the cell, start at the bottom grey triangle. Then if while following a directed path we encounter an element of S, we flip the indicated vertex to move to a new triangle in the cell. For another example for a cell in the hyperbolic group W_{343}, as well as more information about the role of automata in the context of cells, we refer to [11, 12].

For $W = \widetilde{A}_n$, the existence of automata for $\mathtt{Reduced}_W(C)$ follows easily from the work of P. Headley [18] and Shi. Headley proved that one can construct an automaton F recognizing $\mathtt{Reduced}_W$ in which the vertices are the sign-type regions, and in which all vertices are accepting. Hence to recognize $\mathtt{Reduced}_W(C)$ one merely takes F and makes a new automaton F_C by designating only the vertices corresponding to regions in C as accepting. In fact Headley's automaton makes sense for *all* Coxeter groups,[6]

[6]An exposition can be found in Chapter 4 of [8], where F is called the *canonical automaton*

although the examples of $\widetilde{C}_2$ and $\widetilde{G}_2$ already show that the above argument for $\texttt{Reduced}_W(C)$ breaks down. However, for affine Weyl groups, we have conjectured that a closely related automaton works for $\texttt{Reduced}_W(C)$ [17].



(a) (b)

Figure 12.

Further questions

The pictures in this paper certainly raise more questions than they answer. For example, in the case of affine Weyl groups, for all known examples the left cells are of "finite-type," in the sense that they can be encoded by finitely much data. Here we have in mind descriptions of the cells using such tools as patterns among reduced expressions [2, 13, 14], sign-types [29], or similar geometric structures [2, 17].

The cells for general Coxeter groups, on the other hand, appear to be fractal in nature, and thus cannot be described in the same way. Automata provide one convenient way to treat such structures, but they are not the only way. What are other techniques, and which are natural?

The situation becomes even more intriguing when one considers relationships between cells and representation theory. For instance, Lusztig conjectured [24, 3.6] and proved [27] that an affine Weyl group W contains only finitely many two-sided cells. In fact, he proved much more: he showed [28] that there is a remarkable bijection between two-sided cells and the *unipotent conjugacy classes* in the algebraic group dual to that of W. Moreover, each two-sided cell contains only finitely many left cells. Lusztig also conjectured [24, 3.6] that the number of left cells in a two-sided cell can be explicitly given in terms of the cohomology of *Springer varieties* [31].

For general Coxeter groups our knowledge is much more impoverished. First of all, it is not known if there are always only finitely many two-sided cells, although in all known examples it is evidently true. Perhaps the only general result is due to M. Belolipetsky, who showed that *right-angled* hyperbolic Coxeter groups have only 3 two-sided cells [4]. Furthermore, in joint work with M. Belolipetsky we have conjectured that the Coxeter group associated to a hyperbolic n-gon with n distinct angles has $(n+2)$ two-sided cells.

The connection with geometry is even more tenuous. If a Coxeter group W is *crystallographic*, which by definition means $m_{st} \in \{2,3,4,6,\infty\}$ for all distinct generators s,t, then there is associated to W an infinite-dimensional Lie group G called a *Kac-Moody* group. In principle, G provides a setting to study geometric questions about cells, since many of the standard constructions (e.g., flag varieties, Schubert varieties) make sense there. Of course, at the moment the connections with geometry are poorly understood. For instance, the fact that a two-sided cell can contain infinitely many left cells [2–4] is somewhat sobering.

If W is not crystallographic, then there is no such group G. For such W we have no candidate for an algebro-geometric picture. However, computations with many examples (cf. Figures 3 and 4) indicate that certain structures vary "continuously" in families containing both crystallographic and non-crystallographic groups, and that these structures are apparently insensitive to whether or not the underlying group is crystallographic.

The situation is analogous to that of convex polytopes. In the 1980s many difficult theorems about polytopes were first proven using the geometry of certain projective complex varieties—*toric varieties*—built from the combinatorics of *rational* polytopes. Deep properties of the intersection cohomology of these varieties led to highly nontrivial theorems for rational polytopes; for some of these theorems no proofs avoiding geometry were known.

By definition rational polytopes are those whose vertices have rational coordinates. However, not every polytope is rational, and for irrational polytopes no toric variety exists. Yet irrational polytopes seem to share all the nice properties of their rational cousins.

Today we have a much better understanding of this story. Recently several researchers have developed purely combinatorial replacements for the toric variety associated to a rational polytope, and using these replacements have extended various difficult results from the rational case to all polytopes; see [9] for a recent survey of these results.

For Coxeter groups, the analogy suggests developing combinatorial tools to take the role of the algebro-geometric constructions that seem essential in the study of crystallographic groups.[7] Recently there has been significant progress in this effort [15, 16, 30]. Nevertheless, understanding the geometry behind cells for general groups, if it exists, remains an intriguing and difficult problem.

[7] In fact, the analogies between convex polytopes and Coxeter groups go much further than what is suggested in these paragraphs [7], and deserves a lengthy exposition of its own.

References

[1] A. V. Aho, J. E. Hopcroft, and J. D. Ullman, *The design and analysis of computer algorithms*, Addison-Wesley Publishing Co., Reading, Mass.-London-Amsterdam, 1975, Second printing, Addison-Wesley Series in Computer Science and Information Processing.

[2] R. Bédard, *Cells for two Coxeter groups*, Comm. Algebra **14** (1986), no. 7, 1253–1286.

[3] ———, *Left V-cells for hyperbolic Coxeter groups*, Comm. Algebra **17** (1989), no. 12, 2971–2997.

[4] M. Belolipetsky, *Cells and representations of right-angled Coxeter groups*, Selecta Math. (N.S.) **10** (2004), no. 3, 325–339.

[5] R. Bezrukavnikov, *On tensor categories attached to cells in affine Weyl groups*, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 69–90.

[6] R. Bezrukavnikov and V. Ostrik, *On tensor categories attached to cells in affine Weyl groups. II*, Representation theory of algebraic groups and quantum groups, Adv. Stud. Pure Math., vol. 40, Math. Soc. Japan, Tokyo, 2004, pp. 101–119.

[7] A. Björner, *Topological combinatorics*, `www.math.kth.se/ bjorner/files/MacPh60.pdf`, October 2004.

[8] A. Björner and F. Brenti, *Combinatorics of Coxeter groups*, Springer Verlag, 2005.

[9] T. Braden, *Remarks on the combinatorial intersection cohomology of fans*, arXiv:math.CO/0511488.

[10] B. Brink and R. B. Howlett, *A finiteness property and an automatic structure for Coxeter groups*, Math. Ann. **296** (1993), no. 1, 179–190.

[11] W. A. Casselman, *Automata to perform basic calculations in Coxeter groups*, Representations of groups (Banff, AB, 1994), CMS Conf. Proc., vol. 16, Amer. Math. Soc., Providence, RI, 1995, pp. 35–58.

[12] ———, *Regular patterns in Coxeter groups*, talk at CRM, notes avaiable from `www.math.ubc.ca/~cass/crm.talk/toc.html`, January 2002.

[13] C. D. Chen, *The decomposition into left cells of the affine Weyl group of type* $\widetilde{D}_4$, J. Algebra **163** (1994), no. 3, 692–728.

[14] J. Du, *The decomposition into cells of the affine Weyl group of type* $\widetilde{B}_3$, Comm. Algebra **16** (1988), no. 7, 1383–1409.

[15] P. Fiebig, *Kazhdan-Lusztig combinatorics via sheaves on Bruhat graphs*, arXiv:math.RT/0512311.

[16] ———, *The combinatorics of Coxeter categories*, arXiv:math.RT/0512176.

[17] P. E. Gunnells, *On automata and cells in affine Weyl groups*, in preparation.

[18] P. Headley, *Reduced expressions in infinite Coxeter groups*, Ph.D. thesis, University of Michigan, 1994.

[19] J. E. Humphreys, *Representations of reduced enveloping algebras and cells in the affine Weyl group*, arXiv:math.RT/0502100.

[20] ———, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[21] J. C. Jantzen, *Moduln mit einem höchsten Gewicht*, Lecture Notes in Mathematics, vol. 750, Springer-Verlag, Berlin, 1979.

[22] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), no. 2, 165–184.

[23] ———, *Schubert varieties and Poincaré duality*, Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), Proc. Sympos. Pure Math., XXXVI, Amer. Math. Soc., Providence, R.I., 1980, pp. 185–203.

[24] G. Lusztig, *Some examples of square-integrable representations of p-adic semisimple groups*, Trans. Amer. Math. Soc. **277** (1983), 623–653.

[25] ———, *Characters of finite groups of Lie type*, Ann. of Math. Studies, vol. 107, Princeton University Press, 1984.

[26] ———, *Cells in affine Weyl groups*, Algebraic groups and related topics (Kyoto/Nagoya, 1983), Adv. Stud. Pure Math., vol. 6, North-Holland, Amsterdam, 1985, pp. 255–287.

[27] ———, *Cells in affine Weyl groups. II*, J. Algebra **109** (1987), no. 2, 536–548.

[28] ———, *Cells in affine Weyl groups. IV*, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **36** (1989), no. 2, 297–328.

[29] J. Y. Shi, *The Kazhdan-Lusztig cells in certain affine Weyl groups*, Lecture Notes in Mathematics, vol. 1179, Springer-Verlag, Berlin, 1986.

[30] W. Soergel, *Kazhdan-Lusztig-Polynome und unzerlegbare Bimoduln über Polynomringen*, arXiv:math.RT/0403496.

[31] T. A. Springer, *Trigonometric sums, Green functions of finite groups and representations of Weyl groups*, Invent. Math. **36** (1976), 173–207.

[32] D. A. Vogan, Jr., *A generalized τ-invariant for the primitive spectrum of a semisimple Lie algebra*, Math. Ann. **242** (1979), no. 3, 209–224.

Department of Mathematics and Statistics, University of Massachusetts, Amherst, MA 01003

E-mail address: gunnells@math.umass.edu