

COHOMOLOGY OF CONGRUENCE SUBGROUPS OF $SL(4, \mathbb{Z})$ II

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ABSTRACT. In a previous paper [3] we computed cohomology groups $H^5(\Gamma_0(N), \mathbb{C})$, where $\Gamma_0(N)$ is a certain congruence subgroup of $SL(4, \mathbb{Z})$, for a range of levels N . In this note we update this earlier work by extending the range of levels and describe cuspidal cohomology classes and additional boundary phenomena found since the publication of [3]. The cuspidal cohomology classes in this paper are the first cuspforms for $GL(4)$ concretely constructed in terms of Betti cohomology.

1. INTRODUCTION

In this paper we extend the computations in [3] of the cohomology in degree 5 of congruence subgroups $\Gamma_0(N) \subset SL(4, \mathbb{Z})$ with trivial \mathbb{C} coefficients to more levels, up to level 83. We also compute Hecke operators on these cohomology groups and use the Hecke eigenvalues to identify the cohomology eigenclasses as either Eisenstein or cuspidal. We remind the reader that $\Gamma_0(N)$ is the subgroup of $SL(4, \mathbb{Z})$ consisting of matrices whose last row is congruent modulo N to $(0, 0, 0, *)$. We say that $\Gamma_0(N)$ is “modeled” on the $(3, 1)$ parabolic subgroup of $SL(4)/\mathbb{Q}$. Also recall that a cohomological cuspidal automorphic representation contributes to the cohomology of $\Gamma_0(N)$ exactly in degrees 4 and 5.

The size of the matrices and the complexity of computing the Hecke operators are greater the larger N or the more composite N is. Similarly the size of the computation of the Hecke operators at a prime ℓ increases dramatically as a function of both ℓ and N . Therefore after $N = 52$ we stopped computing for composite N but were able to continue for prime N up to level 83. Similarly the size of the computation of the Hecke operators at a prime ℓ increase dramatically with ℓ , so that in fact for the

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new levels in this paper, we computed the Hecke operators only for $\ell = 2$ and in a few cases for $\ell = 3, 5$.

For levels $N = 2$ through 31, we have checked our results by redoing the computations but with coefficients in \mathbb{Z} . In this way, we have also identified non-trivial torsion classes in $H^5(\Gamma_0(N), \mathbb{Z})$ for some levels N . These torsion classes and their relationship to Galois representations will be studied in a future paper.

Working with \mathbb{Z} coefficients is more difficult than with coefficients in a finite field, because the size of the integers in the intermediate steps of the calculations tends to grow exponentially. This is why we stopped using \mathbb{Z} at $N = 31$. For higher levels, we worked over the finite fields $\mathbb{Z}/31991\mathbb{Z}$ or $\mathbb{Z}/12379\mathbb{Z}$.

Unlike the earlier paper, our new results include cuspforms. They also confirm the observed patterns of Eisenstein liftings from the cohomology of the Borel–Serre boundary of the locally symmetric space for $\Gamma_0(N)$ as explained in [3]. We refer the reader to [3] for a detailed explanation of why we look in degree 5, the interaction with the cohomology of the Borel–Serre boundary, and how we perform the computations.

Cuspforms. We discovered cuspforms at levels 61, 73, and 79. All these levels are prime. To the best of our knowledge, these are the first concretely constructed cuspforms for $GL(4)$ in the sense of Betti cohomology. Each cuspform appears with multiplicity two in the cohomology, viewed as a module for the Hecke operators, the eigenvalues being rational integers. In Section 3 we explain this. Theorem 1 asserts that these cuspforms must be functorial liftings from holomorphic Siegel modular forms of weight 3 on $GSp(4)/\mathbb{Q}$.

It would be interesting to see a construction of these Siegel modular forms. There are several potential approaches. One is to construct them using theta series. Such theta series would be on the congruence subgroups modeled on the Klingen parabolic subgroup in $GSp(4)/\mathbb{Q}$. Unfortunately all the work we know of that might have been relevant, for example [12], concerns congruence subgroups modeled on the Siegel parabolic subgroup. These latter theta series lead to Siegel modular forms that can appear in the cohomology of the congruence subgroups of $SL(4, \mathbb{Z})$ modeled on the $(2, 2)$ parabolic subgroup of $SL(4)/\mathbb{Q}$. We plan to investigate the cohomology of such subgroups in future work. Another possible construction of the desired Siegel modular forms is by computing the cohomology of congruence subgroups of $Sp(4, \mathbb{Z})$, since holomorphic Siegel modular forms of weight 3 will contribute to the cohomology of these groups. Finally, one could also try to isolate the motives corresponding to the cuspidal automorphic forms we found. These might be found either as factors in the cohomology of the appropriate Siegel modular variety, or by other means, as in the work of van Geemen and Top [7].

We remark that Ibukiyama [11] conjectured (and has recently announced a proof of) a formula that describes the dimensions of weight three cuspidal Siegel modular forms on the paramodular groups of prime level. The paramodular group of level N

is a congruence subgroup of $Sp(4, \mathbb{Q})$ that contains the congruence subgroup of level N based on the Klingen parabolic subgroup. Gritsenko [9] has constructed a lift from Jacobi forms to Siegel modular forms on the paramodular group. Brumer observed that the first levels where the forms predicted by Ibukiyama are not accounted for by Gritsenko's lifts are 61, 73, and 79, and in these cases the subspace of lifts has codimension one. Thus we expect that our classes will prove to be concrete realizations of the lifts of these Siegel modular forms.

It would be most interesting to discover cohomology classes in $H^5(\Gamma_0(N), \mathbb{C})$ corresponding to cuspforms that are not lifts from any smaller group, but these have not shown up yet in our computations. Each such cuspidal Hecke eigenclass would give rise to a 4-dimensional subspace of the cohomology over \mathbb{C} , namely two subspaces each of multiplicity two, with Hecke eigenvalues the complex conjugates of each other (see Section 3).

Eisenstein series. We continue to observe that all weight 2 newforms for $GL(2)/\mathbb{Q}$ of level dividing N lift as cohomology of the Borel–Serre boundary into the cohomology of our $\Gamma_0(N)$. Only certain weight 4 newforms f were observed to lift. We conjecture that such a form lifts if and only if the sign in its functional equation is negative (see Conjecture 1 below for more details). The connection between this sign and our observed lifting phenomenon was pointed out to us by U. Weselmann.

For prime levels N , we have also observed that cohomology classes with trivial coefficients of level N attached to cuspidal automorphic representations of $GL(3)/\mathbb{Q}$ lift to $H^5(\Gamma_0(N), \mathbb{C})$, again via the cohomology of the Borel–Serre boundary.

When the level N is a square, we observed that some cohomology for minimal faces of the Borel–Serre boundary lifts to H^5 . The details of this phenomenon are currently unclear.

Eisenstein cohomology, originally introduced by G. Harder (cf. [10]), has been investigated extensively by J. Franke, J. Rohlfs, J. Schwermer, B. Speh, and others. However, it is a difficult open problem to compute in all detail the cohomology of the Borel–Serre boundary for $GL(4)$ for general N . Even if that were done, current results in Eisenstein cohomology don't appear to be fine enough even to check Conjecture 1, which is only for prime level. One problem is that trivial coefficients is much harder to handle than irreducible coefficient modules with regular highest weights.

In the final section of the paper we provide some tables of results. Table 1 shows the new Betti numbers we computed and extends the data in [3].¹ Table 2 shows the Hecke polynomials for the Eisenstein classes we found that are covered by Conjecture 1. Finally Table 3 gives the Hecke polynomials and eigenvalues for the cuspidal classes we found.

¹Level 49 was incorrectly reported in [3]. Levels 55, 67, 71 were conjectured in [3].

2. EISENSTEIN CLASSES

Let $\xi \in H^5(\Gamma_0(N), \mathbb{C})$ be a Hecke eigenclass. Recall [3, §1.1] that for us this means ξ is an eigenvector for certain operators

$$T(l, k): H^5(\Gamma_0(N), \mathbb{C}) \rightarrow H^5(\Gamma_0(N), \mathbb{C}),$$

where $k = 1, 2, 3$ and l is a prime not dividing N . These operators correspond to the double cosets $\Gamma_0(N)D(l, k)\Gamma_0(N)$, where $D(l, k)$ is the diagonal matrix with $4 - k$ ones followed by k l 's. Suppose the eigenvalue of $T(l, k)$ on ξ is $a(l, k)$, and define the Hecke polynomial $H(\xi)$ of ξ by

$$H(\xi) = \sum_k (-1)^k l^{k(k-1)/2} a(l, k) T^k \in \mathbb{C}[T].$$

Let $\rho: \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_n(\mathbb{Q}_p)$ be a continuous semisimple Galois representation unramified outside pN . Then we say the eigenclass ξ is *attached* to ρ if for all l not dividing pN we have

$$H(\xi) = \det(1 - \rho(\text{Frob}_l)T).$$

The goal of this section is to formulate a conjecture about some Eisenstein classes in the cohomology $H^5(\Gamma_0(N), \mathbb{C})$. We recall the definition; for more background on Eisenstein cohomology and its applications, we refer to [10].

Let X be the global symmetric space $SL(4, \mathbb{R})/SO(4)$, and let X^{BS} be the partial compactification constructed by Borel and Serre [6]. The quotient $Y := \Gamma_0(N) \backslash X$ is an orbifold, and the quotient $Y^{\text{BS}} := \Gamma_0(N) \backslash X^{\text{BS}}$ is a compact orbifold with corners. We have $H^*(\Gamma_0(N), \mathbb{C}) \simeq H^*(Y, \mathbb{C}) \simeq H^*(Y^{\text{BS}}, \mathbb{C})$.

Let $\partial Y^{\text{BS}} = Y^{\text{BS}} \setminus Y$. The Hecke operators act on the cohomology of the boundary $H^*(\partial Y^{\text{BS}}, \mathbb{C})$, and the inclusion of the boundary $\iota: \partial Y^{\text{BS}} \rightarrow Y^{\text{BS}}$ induces a map on cohomology $\iota^*: H^*(Y^{\text{BS}}, \mathbb{C}) \rightarrow H^*(\partial Y^{\text{BS}}, \mathbb{C})$ compatible with the Hecke action. The kernel $H_!^*(Y^{\text{BS}}, \mathbb{C})$ of ι^* is called the *interior cohomology*; it contains the cohomology with compact supports. The goal of Eisenstein cohomology is to use Eisenstein series and cohomology classes on the boundary to construct a Hecke-equivariant section $s: H^*(\partial Y^{\text{BS}}, \mathbb{C}) \rightarrow H^*(Y^{\text{BS}}, \mathbb{C})$ mapping onto a complement $H_{\text{Eis}}^*(Y^{\text{BS}}, \mathbb{C})$ of the interior cohomology in the full cohomology. We call classes in the image of s *Eisenstein classes*. (In general, residues of Eisenstein series can give interior, noncuspidal cohomology classes, with infinity type a Speh representation, but as noted in [3], these do not contribute to degree 5.)

To describe our conjectural Eisenstein classes, we give the Galois representations we believe are attached to the classes along with the corresponding Hecke polynomials. In the following, ε denotes the p -adic cyclotomic character, so that $\varepsilon(\text{Frob}_l) = l$ for any prime l coprime to p . We denote the trivial representation by i .

- *Weight two holomorphic modular forms:* Let σ_2 be the Galois representation attached to a holomorphic weight 2 newform f of level N with trivial Nebentypus. Let α be the eigenvalue of the classical Hecke operator T_l on f . Let $\text{IIa}(\sigma_2)$ and $\text{IIb}(\sigma_2)$ be the Galois representations in the first two rows of Table 2 (see p. 12).
- *Weight four holomorphic modular forms:* Let σ_4 be the Galois representation attached to a holomorphic weight 4 newform f of level N with trivial Nebentypus. Let β be the eigenvalue of the classical Hecke operator T_l on f . Let $\text{IV}(\sigma_4)$ be the Galois representation in the third row of Table 2.
- *Cuspidal cohomology classes from subgroups of $SL(3, \mathbb{Z})$:* Let τ be the Galois representation conjecturally attached to a pair of nonselfdual cuspidal cohomology classes $\eta, \eta' \in H^3(\Gamma_0^*(N), \mathbb{C})$, where $\Gamma_0^*(N) \subset SL(3, \mathbb{Z})$ is the congruence subgroup with bottom row congruent to $(0, 0, *)$ modulo N . Let γ be the eigenvalue of the Hecke operator $T_{l,1}$ on η , and let γ' be its complex conjugate. Let $\text{IIIa}(\tau)$ and $\text{IIIb}(\tau)$ be the Galois representations in the last two rows of Table 2.

If f is a weight 2 or weight 4 eigenform as above, we denote by d_f the degree of the extension of \mathbb{Q} generated by the eigenvalues of f . Similarly for an eigenclass $\eta \in H^3(\Gamma_0^*(N), \mathbb{C})$ we write d_η for the degree of the field generated by the eigenvalues of η . Also, the L -function $\Lambda(f, s)$ of a holomorphic modular form of even weight k and level N refers to the function

$$\Lambda(f, s) = N^{s/2} (2\pi)^{-s} \Gamma(s) L(f, s),$$

where $L(f, s)$ is the Dirichlet series for f . The function $\Lambda(f, s)$ satisfies the functional equation

$$\Lambda(f, s) = w(-1)^{k/2} \Lambda(f, k - s),$$

where $w \in \{\pm 1\}$ gives the action of the Fricke involution.

Conjecture 1. *Fix a positive prime p . Then the cohomology group $H^5(\Gamma_0(p), \mathbb{C})$ contains the following Eisenstein subspaces:*

- (1) *For each weight two holomorphic newform f of level p with associated Galois representation σ_2 , two d_f -dimensional subspaces, one attached to the Galois representation $\text{IIa}(\sigma_2)$, and the other to the Galois representation $\text{IIb}(\sigma_2)$.*
- (2) *For each weight four holomorphic newform f of level p with associated Galois representation σ_4 such that the sign w of the functional equation of the L -function $\Lambda(f, s)$ is negative, a d_f -dimensional subspace attached to the Galois representation $\text{IV}(\sigma_2)$.*
- (3) *For each pair of nonselfdual cuspidal cohomology class $\eta, \eta' \in H^3(\Gamma_0^*(p), \mathbb{C})$, $\Gamma_0^*(p) \subset SL(3, \mathbb{Z})$ with conjecturally associated Galois representation τ , two d_η -dimensional subspaces, one attached to the Galois representation $\text{IIIa}(\tau)$, and the other to the Galois representation $\text{IIIb}(\tau)$.*

Example 1. Let $N = 53$. Then N is prime, and is in fact the first level for which $H^3(\Gamma_0^*(N), \mathbb{C})$ contains nontrivial cuspidal classes. According to [2], there are two nonselfdual cuspidal classes η, η' whose Hecke eigenvalues are complex conjugates of each other. Moreover if $T(l, 1)\eta = a(l, 1)\eta$, then one knows that $T(l, 2)\eta = \bar{a}(l, 1)\eta$. Writing $\omega = (1 + \sqrt{-11})/2$, the Hecke eigenvalues of η are given by the following table:

l	2	3	5	7	11	13
$a(l, 1)$	$-1 - 2\omega$	$-2 + 2\omega$	1	-3	1	$-2 - 12\omega$

According to Conjecture 1, this pair should contribute a 4-dimensional subspace to $H^5(\Gamma_0(53), \mathbb{C})$.

Now we consider the subspaces corresponding to modular forms. By consulting tables of modular forms of weights 2 and 4 [17], we find (i) the dimension of the space of weight 2 newforms of level 53 is 4, with one form rational and one defined over the real cubic field of discriminant 148, and (ii) the dimension of the space of weight 4 newforms of level 53 is 13, with one form rational, one defined over a real quartic field, and one defined over a real octic field. We can also see that the rational and quartic weight 4 forms have negative sign in their functional equations. Hence the weight 2 forms should contribute a $2(1 + 3) = 8$ dimensional subspace, while the weight 4 forms should contribute a $1 + 4 = 5$ dimensional subspace.

Thus the final result predicted by Conjecture 1 is that $\dim H^5(\Gamma_0(53), \mathbb{C}) \geq 4 + 8 + 5 = 17$. Indeed, our computations show $\dim H^5(\Gamma_0(53), \mathbb{C}) = 17$, and that the Hecke polynomials of the eigenclasses match those predicted by Conjecture 1 at $l = 2$. In particular, there is no cuspidal cohomology at level 53.

We remark that if we allow N to be composite, we know that Conjecture 1 does not give a complete description of the Eisenstein subspace of $H^5(\Gamma_0(N), \mathbb{C})$. For instance, the cohomology contains Eisenstein classes corresponding to minimal faces of the Borel–Serre boundary when N is a square, as mentioned on p. 3. Moreover, newforms for levels properly dividing N also appear in H^5 . As an example, it appears from our data that if a prime p exactly divides N , then a weight 2 newform f at level p contributes two $3d_f$ -dimensional subspaces to $H^5(\Gamma_0(N), \mathbb{C})$, corresponding to three copies of the representations IIa and IIb. Similar phenomena occur for weight 4 forms. Also, for sufficiently composite levels, Eisenstein cohomology involving weight 3 forms with odd character can occur, as happens e.g. for level 50. Since we don't fully understand the mechanisms underlying these lifts, we restrict Conjecture 1 to prime level.

3. CUSPFORMS

Let π be a cuspidal automorphic representation for $GL(4, \mathbb{A})$, where \mathbb{A} denotes the adèle group of \mathbb{Q} . Assume that π contributes to the cohomology $H^5(\Gamma_0(N), \mathbb{C})$. Then the infinity type π_∞ is uniquely determined, and is denoted π_1 in the table

of [1, p. 65]. As explained there, when restricted to $SL(4, \mathbb{R})$, π_1 breaks up into the direct sum $\pi_1^+ \oplus \pi_1^-$, whose components are interchanged by the inner automorphism ι induced by $\text{diag}(-1, 1, 1, 1)$.

It follows from this last fact that $H_{\text{cusp}}^5(\Gamma_0(N), \mathbb{C})$, the cuspidal part of the cohomology, will have isotypic components for the action of the Hecke algebra of even dimension $2k$, and ι will act as an involution on each isotypic component interchanging two complementary subspaces of dimension k . (Compare what happens for $GL(2)$, where every cohomological cuspidal automorphic representation contributes to the group cohomology twice, once as a holomorphic modular form and once as an anti-holomorphic form.)

Let f be a Hecke eigenclass in $H_{\text{cusp}}^5(\Gamma_0(N), \mathbb{C})$. For a fixed prime ℓ not dividing N , let a, b, c be the Hecke eigenvalues of $T_{\ell,1}, T_{\ell,2}, T_{\ell,3}$ respectively. Then the Hecke polynomial at ℓ is by definition

$$P(X) = 1 - aX + b\ell X^2 - c\ell^3 X^3 + \ell^6 X^4.$$

Suppose π is the cuspidal automorphic representation associated to f . We refer to [4, pp. 756–7] for the following facts. Letting c denote complex conjugation, there is defined another (or possibly the same) cuspidal automorphic representation ${}^c\pi$ with the property that the Hecke eigenvalues of the corresponding cohomology class ${}^c f$ are $\bar{a}, \bar{b}, \bar{c}$.

There is also the contragredient cuspidal automorphic representation $\tilde{\pi}$ and the Hecke eigenvalues of its corresponding cohomology class \tilde{f} are c, b, a . Because the coefficients of our cohomology class are trivial, the weight w in the notation of [4] equals 3 and ${}^c\pi = \tilde{\pi}$. Therefore $a = \bar{c}$ and $b = \bar{b} \in \mathbb{R}$. We say that f or π are *selfdual* if $\tilde{\pi} \simeq \pi$. This happens if and only if $a = c$ and hence if and only if $a, b, c \in \mathbb{R}$.

One can recognize nonzero elements in $H_{\text{cusp}}^5(\Gamma_0(N), \mathbb{C})$ as follows. Compute Hecke eigenvalues on the whole cohomology $H^5(\Gamma_0(N), \mathbb{C})$. Any system of eigenvalues that does not appear to be attached to an Eisenstein cohomology class must be attached to cuspidal cohomology. For example, if even a single Hecke polynomial is irreducible, then the corresponding Hecke eigenspace must be cuspidal. A further check is given by the fact that a cuspidal eigenspace must be even dimensional.

Let V be a minimal nontrivial Galois-stable Hecke eigenspace in $H_{\text{cusp}}^5(\Gamma_0(N), \overline{\mathbb{Q}})$. The Hecke eigenvalues on V generate an order R in the ring of integers in some number field. If R is totally real then the corresponding automorphic representations are self-dual and should be functorial liftings from a smaller group that fixes a quadratic form, i.e. either from $GSp(4)$ or $GO(4)$. Otherwise, R must generate a complex CM field (see [4]).

For more information about the situation where R is totally real, in which the desired results are close to being proved, we refer to [13, 16]. In brief, assume π is essentially selfdual, i.e. $\tilde{\pi}$ is isomorphic to $\pi \otimes \chi$ for some character χ . The L -groups of $GSp(4)$ and $GO(4)$ can be identified, respectively, with their complex points. Then

π should descend to a cusp form Π on $GO(4)/\mathbb{Q}$ (resp. $GSp(4)/\mathbb{Q}$) iff the symmetric square (resp. the exterior square) L-function of π admits, when twisted by the inverse of a character ν (“similitude norm”), a pole at $s = 1$. (This corresponds to the symmetric (resp. exterior) square of the associated 4-dimensional representation σ of the conjectural Langlands group $\mathcal{L}_{\mathbb{Q}}$ having a stable line.)

We found two-dimensional spaces of cuspidal cohomology at levels $N = 61, 73, 79$, and in each case the Hecke polynomial at 2 was irreducible. The rest of the cohomology at these levels is accounted for by the Eisenstein subspaces of Conjecture 1. Since V in each of these cases is 2-dimensional, the Hecke eigenvalues in each case must be rational integers. Therefore these cuspforms are selfdual and are expected to be lifts from $GSp(4)$ or $GO(4)$.

If we assume the Weil bounds for our Hecke eigenvalues, they tell us that the eigenvalues a, c at ℓ all have absolute value less than or equal to $4\ell^{3/2}$ and $|b| \leq 6\ell^2$. Hence although we only work modulo 31991 or 12379, we can assert the eigenvalues as found in Table 3. For level 79, we encountered overflow errors working modulo 31991, and instead we worked modulo 12379.

If the Hecke eigenvalues were to generate for example an imaginary quadratic extension of \mathbb{Q} , V would have to be at least 4 dimensional and the corresponding cuspforms would not be lifts from smaller groups. It would be of great interest to find examples of this. The analogous objects do exist for $GL(3)$ as first found in [2].

As stated above, the cuspforms we found are expected to be lifts from $GSp(4)$ or $GO(4)$. In fact, thanks to an argument shown us by Ramakrishnan, we can show that for these levels, our cuspforms are always lifts from $GSp(4)$:

Theorem 1. *Let f be a Hecke eigenclass in $H_{cusp}^5(\Gamma_0(N), \mathbb{C})$ and π the cuspidal automorphic representation associated to f . Assume that π is a functorial lift from $GO(4)$. Then N cannot be squarefree.*

Proof. (D. Ramakrishnan) The archimedean parameter of π is a homomorphism

$$\sigma_{\infty} : W_{\mathbb{R}} \rightarrow GL(4, \mathbb{C}),$$

where $W_{\mathbb{R}}$ is the real Weil group containing \mathbb{C}^* as a subgroup and $\text{Gal}(\mathbb{C}/\mathbb{R})$ as the corresponding quotient, whose non-trivial element c acts on \mathbb{C}^* by sending z to its complex conjugate \bar{z} . Since π contributes to cuspidal cohomology with constant coefficients of the congruence subgroup Γ of $SL(4, \mathbb{Z})$, one necessarily has (in the unitary normalization):

$$\sigma_{\infty} \simeq I(W_{\mathbb{R}}, \mathbb{C}^*; \alpha^3) \oplus I(W_{\mathbb{R}}, \mathbb{C}^*; \alpha),$$

where I denotes induction, here from \mathbb{C}^* to $W_{\mathbb{R}}$, and $\alpha = z/|z|$. Consequently the restriction of σ_{∞} to \mathbb{C}^* is the sum of the characters in the “infinity type”:

$$p_{\infty} = \{\alpha^3, \alpha, \alpha^{-1}, \alpha^{-3}\}.$$

Suppose π is of general orthogonal type. Then it is either of the following two kinds:

$$(I) \quad \pi = \pi_1 \boxtimes \pi_2,$$

where π_1, π_2 are cusp forms on $GL(2)/\mathbb{Q}$ and \boxtimes is the Rankin–Selberg (or automorphic) tensor product, which corresponds to the tensor product (not the direct sum) of the corresponding 2-dimensional representations of $\mathcal{L}_{\mathbb{Q}}$; or

$$(II) \quad \pi = \text{As}_{K/\mathbb{Q}}(\eta),$$

where π is the Asai representation defined by a cusp form η of $GL(2)/K$, where K is a quadratic extension of \mathbb{Q} .

In case (I), suppose one of the π_j , say π_2 , is an Eisenstein class of the form $\mu_1 \boxplus \mu_2$ (“isobaric sum”). This means $L(s, \pi_2) = L(s, \mu_1)L(s, \mu_2)$, which implies $\pi = (\pi_1 \otimes \mu_1) \boxplus (\pi_1 \otimes \mu_2)$, and thus π is certainly not cuspidal.

Continuing with case (I), assume now that both the π_j are cuspidal and let $\sigma_{j, \infty}$ denote the $W_{\mathbb{R}}$ -parameter of the cusp form π_j . Since the two irreducible constituents of σ_{∞} are not twist equivalent, $\sigma_{1, \infty}$ and $\sigma_{2, \infty}$ are both forced to be irreducible. We may write, after possibly interchanging π_1 and π_2 ,

$$\sigma_{1, \infty} = I(W_{\mathbb{R}}, \mathbb{C}^*; \alpha^a)$$

and

$$\sigma_{2, \infty} = I(W_{\mathbb{R}}, \mathbb{C}^*; \alpha^b)$$

with $a \geq b > 0$.

Since the Rankin–Selberg product $(\pi_1, \pi_2) \rightarrow \pi$ is functorial at all places [15], in particular at ∞ , we must have

$$\sigma_{\infty} = \sigma_{1, \infty} \otimes \sigma_{2, \infty},$$

implying that

$$a + b = 3, a - b = 1,$$

so that $a = 2, b = 1$. In other words, π_1, π_2 are classical holomorphic newforms of weight 3, 2 respectively. Let N_j be the level (conductor) of π_j (for $j = 1, 2$) and N the conductor of π . We have

$$(N_1, N_2) = 1 \implies N = N_1^2 N_2^2$$

(see [5]). Since we are assuming N is squarefree, this cannot happen.

More generally, if at any prime p , $v_p(N_1) > 0$ and $v_p(N_2) = 0$, then $v_p(N) = 2v_p(N_1)$. So for N to be square-free, it is necessary that $v_p(N_1) = v_p(N_2)$ at all p . In other words, $v_p(N_1)$ and $v_p(N_2)$ are simultaneously 0 or simultaneously 1.

Now let p divide N . It is then the case that up to an unramified twist that can be ignored, $\pi_{1,p} \simeq \pi_{2,p} \simeq St_p$. (See, for example, the table on p. 73 of [8] giving

the conductors of representations of $GL(2, \mathbb{Q}_p)$; note also that the conductor of any ramified twist of St_p is divisible by p^2 .) The reason is that

$$St_p \boxtimes St_p = \text{sym}^2(St_p) \boxplus 1,$$

where $\text{sym}^2(St_p)$, being the Steinberg representation of $GL(3, \mathbb{Q}_p)$, is also of conductor p . Indeed, St_p corresponds, by the local correspondence at p , to

$$\tau_p = 1 \otimes id : W_{\mathbb{Q}_p} \times SL(2, \mathbb{C}) \rightarrow GL(2, \mathbb{C}), (w, g) \rightarrow g,$$

and the Steinberg representation of $GL(3, \mathbb{Q}_p)$ corresponds to

$$1 \otimes \text{sym}^2 : W_{\mathbb{Q}_p} \times SL(2, \mathbb{C}) \rightarrow GL(3, \mathbb{C}).$$

The only other possible representations of $GL(2, \mathbb{Q}_p)$ of conductor p are the principal series representation $\xi_1 \boxplus \xi_2$, with ξ_1 of conductor p and ξ_2 unramified. But if $\pi_{2,p}$ is of this form with $\pi_{1,p} = St_p$, the conductor of π_p will be divisible by p^2 . Similarly, if $\pi_{1,p}$ and $\pi_{2,p}$ are both principal series of conductor p , then the conductor of π_p will be divisible by p^2 .

In our case, π_1 is generated by a holomorphic newform of weight 3, hence has a non-trivial character since its character must have the same parity as the weight of π_1 , and hence must be odd. This character must then be ramified at some prime p_0 , say, because \mathbb{Q} has class number 1. This p_0 must divide N , and we get a contradiction from π_{1,p_0} being St_{p_0} , which has trivial central character. Note that this argument depends on N being squarefree.

Now we can move to case (II). Suppose $\pi = \text{As}_{K/\mathbb{Q}}(\eta)$, for a cusp form η on $GL(2)/K$, K a quadratic field. A basic property of Asai representations gives [14]

$$\pi_K = \eta \boxtimes (\eta^\theta),$$

where π_K denotes the base change of π to $GL(4)/K$, θ is the nontrivial automorphism of K , and η^θ means $\eta \circ \theta$. Let τ_∞ denote the 2-dimensional representation of W_{K_∞} associated to η_∞ , so that τ_∞^θ is associated to η_∞^θ .

First consider the case of a real quadratic K . Then we have

$$\sigma_\infty = \tau_\infty \otimes \tau_\infty^\theta.$$

Arguing as in case (I) we see that τ_∞ should have parameter $\{\alpha^2, \alpha^{-2}\}$, while τ_∞^θ has parameter $\{\alpha, \alpha^{-1}\}$, which is clearly impossible.

So we may assume that K is imaginary quadratic where θ induces complex conjugation on $K_\infty = \mathbb{C}$. Suppose the archimedean parameter of η is $\{\alpha^a, \alpha^c\}$. Since $W_{\mathbb{C}} = \mathbb{C}^*$, this need not be preserved by complex conjugation. But nevertheless, it forces the archimedean parameter of η^θ to be $\{\alpha^{-a}, \alpha^{-c}\}$, and the tensor product of these two parameters is not regular, though tempered, and thus cannot contribute to the cohomology of an arithmetic group. \square

We remark that from [1, p. 52] we see that the automorphic representation on $GSp(4)$ that we lift to get our f must correspond to a holomorphic Siegel modular form of weight 3.

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TABLES OF RESULTS

Level	rank	Level	rank	Level	rank
2	0	23	5	44	18
3	0	24	2	45	27
4	0	25	7	46	19
5	0	26	7	47	11
6	0	27	12	48	26
7	0	28	7	49	20
8	0	29	6	50	34
9	3	30	8	51	19
10	0	31	6	52	21
11	2	32	12	53	17
12	0	33	10	54	49
13	1	34	12	55	15
14	2	35	7	56	20
15	2	36	24	57	19
16	3	37	8	59	14
17	3	38	14	61	20
18	9	39	10	67	17
19	3	40	9	71	17
20	2	41	9	73	20
21	3	42	17	79	25
22	7	43	10	83	21

TABLE 1. Betti numbers for $H^5(\Gamma_0(N), \mathbb{C})$.

IIa	$\sigma_2 \oplus \varepsilon^2 \oplus \varepsilon^3$	$(1-l^2T)(1-l^3T)(1-\alpha T+lT^2)$
IIb	$i \oplus \varepsilon^2 \sigma_2 \oplus \varepsilon$	$(1-T)(1-lT)(1-l^2\alpha T+l^5T^2)$
IV	$\sigma_4 \oplus \varepsilon \oplus \varepsilon^2$	$(1-lT)(1-l^2T)(1-\beta T+l^3T^2)$
IIIa	$\tau \oplus \varepsilon^3$	$(1-l^3T)(1-\gamma T+l\gamma'T^2-l^3T^3)$
IIIb	$i \oplus \varepsilon\tau$	$(1-T)(1-l\gamma T+l^3\gamma'T^2-l^6T^3)$

TABLE 2. Galois representations and Hecke polynomials for Eisenstein classes. See Section 2 for explanation of notation.

Level 61		
T_2	$(-7, 12, -7)$	$1 + 7T + 24T^2 + 56T^3 + 64T^4$
T_3	$(-3, 1, -3)$	$1 + 3T + 3T^2 + 81T^3 + 729T^4$
Level 73		
T_2	$(-6, 11, -6)$	$1 + 6T + 22T^2 + 48T^3 + 64T^4$
T_3	$(-2, 1, -2)$	$1 + 2T + 3T^2 + 54T^3 + 729T^4$
Level 79		
T_2	$(-5, 7, -5)$	$1 + 5T + 14T^2 + 40T^3 + 64T^4$
T_3	$(-5, 14, -5)$	$1 + 5T + 42T^2 + 135T^3 + 729T^4$

TABLE 3. Eigenvalues and Hecke polynomials for $H_{\text{cusp}}^5(\Gamma_0(N), \mathbb{C})$.

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