MOD 2 HOMOLOGY FOR $GL(4)$ AND GALOIS REPRESENTATIONS

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Dedicated to the memory of Steve Rallis

Abstract. We extend the computations in [AGM11] to find the mod 2 homology in degree 1 of a congruence subgroup Γ of $\mathrm{SL}(4,\mathbb{Z})$ with coefficients in the sharbly complex, along with the action of the Hecke algebra. This homology group is closely related to the cohomology of Γ with $\mathbb{F}_2$ coefficients in the top cuspidal degree. These computations require a modification of the algorithm to compute the action of the Hecke operators, whose previous versions required division by 2. We verify experimentally that every mod 2 Hecke eigenclass found appears to have an attached Galois representation, giving evidence for a conjecture in [AGM11]. Our method of computation was justified in [AGM12].

1. Introduction

1.1. This is a continuation of a series of papers [AGM02, AGM08, AGM10, AGM11, AGM12] devoted to the computation of the cohomology of congruence subgroups $\Gamma \subset \mathrm{SL}(4,\mathbb{Z})$ with constant coefficients, together with the action of the Hecke operators on the cohomology. We also investigated the representations of the absolute Galois group of $\mathbb{Q}$ that appear to be attached to Hecke eigenclasses in the cohomology. The papers [AGM02, AGM08, AGM10] deal with complex coefficients, while [AGM11, AGM12] deal with coefficients in a prime finite field $\mathbb{F}_p$, with p odd. The current paper takes $p = 2$. We concentrate on $H^5(\Gamma)$ because on the one hand, H^5 supports cuspidal cohomology with $\mathbb{C}$-coefficients, and on the other hand it is only one degree below the virtual cohomological dimension of Γ and therefore amenable to an algorithm due to one of us (PG) for computing Hecke operators [Gun00]. Our next project will be to rewrite our code to deal with finite-dimensional twisted coefficients, which should lead to more interesting examples of attached Galois representations, aimed at testing the generalization of Serre's conjecture found in [ADP02] and in [Her09].

As explained in [AGM11], when $p > 5$, the $\mathbb{C}$- and mod p-betti numbers coincide. In this case we can compute the cohomology in terms of the Steinberg module and the sharbly complex, which is what we did. Namely, $H^5(\Gamma, K) \approx H_1(\Gamma, St \otimes K) \approx$

1991 *Mathematics Subject Classification.* Primary 11F75; Secondary 11F67, 20J06, 20E42.

Key words and phrases. Cohomology of arithmetic groups, Galois representations, Voronoi complex, Steinberg module, modular symbols.

AA wishes to thank the National Science Foundation for support of this research through NSF grant DMS-0455240, and also the NSA through grant H98230-09-1-0050. This manuscript is submitted for publication with the understanding that the United States government is authorized to produce and distribute reprints. PG wishes to thank the National Science Foundation for support of this research through NSF grant DMS-0801214 and DMS-1101640.

$H_1(\Gamma, Sh_\bullet \otimes K)$. Here, $K = \mathbb{C}$ or $\mathbb{F}_p$, $\Gamma \subset \mathrm{SL}(4,\mathbb{Z})$ is a congruence subgroup, St denotes the Steinberg module, and $Sh_\bullet$ the sharbly complex, whose definitions are recalled in Section 2 below.

What we actually compute is the homology valued in the sharbly complex, because only in the sharbly complex do we know how to compute the Hecke action in a feasible way. In theory we could compute the mod p cohomology using a spectral sequence similar to that used by Soulé for $\mathrm{SL}(3,\mathbb{Z})$ in [Sou78]. However, even if we carried out this arduous task, we do not know how to compute the Hecke action on the resulting cohomology.

The method we use to compute $H_1(\Gamma, Sh_\bullet \otimes \mathbb{F}_2)$ is the same as in the previous papers. However, the algorithm in [Gun00] for the computation of the Hecke action had required division by 2, which prevented our treatment of mod 2 coefficients. Following a suggestion of Dan Yasaki, we overcome that problem in this paper.

1.2. The mod 2 homology is especially interesting for two reasons. One is that there are many more mod 2 classes than exist for odd primes, so there is more opportunity for testing conjectures and studying phenomenology. The other is that every mod 2 Galois representation is odd and therefore again there are more possibilities for investigating the Serre-type conjectures.

By Theorem 13 of [AGM12], the Hecke eigenvalue data we compile gives us parts of Hecke eigenpackets ocurring in the sharbly homology of $\Gamma_0(N)$, for various N. Therefore we can test Conjecture 5(d) of [AGM11], which asserts the existence of a Galois representation unramified outside $2N$ associated to each such eigenpacket. We do this by searching for the Galois representation using a computer program described in Section 4. There can easily be more than one Galois representation that fits our data for any given Hecke eigenclass, because we have only computed a few Hecke operators at each level (because of time and space constraints). Our Galois finder searches for the "simplest" Galois representation that fits our data in each case. We use the supply of characters and 2-dimensional representations coming from classical modular forms of weights 2, 3 and 4. In no case do we fail to find a match, using just reducible representations made out of these blocks. An explanation of why we use just these weights appears in Section 4.

Although we stop searching when we have found one Galois representation that appears to be attached to a given Hecke eigenpacket, we know by the Brauer-Nesbitt Theorem that up to semisimplification there can be at most one Galois representation that is truly attached. This Galois representation might be describable in many different ways using characters and classical cuspforms, because such things can be congruent modulo a prime above 2. Of course, we would expect more complicated and even irreducible 4-dimensional representations to be needed if we could compute for much larger levels and more Hecke operators. But at least in this small way we find evidence both for Conjecture 5(d) of [AGM11] and of the correctness of our computations.

1.3. We now give a guide to the paper. In Section 2 we recall the definitions of the Steinberg module, the sharbly complex, and the concept of attached Galois representation. We state the conjecture of [AGM11] that asserts the existence of attached Galois representations to Hecke eigenclasses in the sharbly homology.

In Section 3 we describe what we actually compute, namely certain Hecke eigenclasses in the sharbly homology in degree 1. We use the Voronoi complex. We describe how the sharbly homology is calculated as a Hecke module, with reference to our earlier papers for details. Then we explain what modifications we made to the Hecke algorithm to allow us to work with $\mathbb{F}_2$-coefficients.

In Section 4 we describe our Galois representation finder. Because there are so many mod 2 homology classes, we had to automate the process of finding candidates for the conjecturally attached Galois representations.

In Section 5 we give our results. We give the level N of Γ, the dimension of $H_1(\Gamma, Sh_\bullet \otimes \mathbb{F}_2)$, and an enumeration of packets of Hecke eigenvalues. For each packet, we give the dimension of its simultaneous eigenspace and a Galois representation that appears to be attached to the packet.

It may be seen that every Galois representation that appears in our results is reducible. Of course, if the conjecture of [ADP02] is true, and if there is a robust enough connection between sharbly homology and group cohomology, then there should exist plenty of irreducible Galois representations attached to sharbly homology eigenclasses. But they would occur at levels far too large for us to compute.

We thank Dan Yasaki for conversations that greatly helped this project at the start. We thank Kevin Buzzard for very helpful correspondence, particularly in regard to (4.8).

2. The Steinberg module and the sharbly complex

2.1. Let $n \geqslant 2$ and let $\mathbb{Q}^n$ denote the vector space of n-dimensional row vectors.

2.2. Definition. The *Sharbly complex* $Sh_\bullet$ is the complex of $\mathbb{Z}\mathrm{GL}(n, \mathbb{Q})$-modules defined as follows. As an abelian group, Sh_k is generated by symbols $[v_1, \ldots, v_{n+k}]$, where the v_i are nonzero vectors in $\mathbb{Q}^n$, modulo the submodule generated by the following relations:

(i) $[v_{\sigma(1)}, \ldots, v_{\sigma(n+k)}] - (-1)^\sigma [v_1, \ldots, v_{n+k}]$ for all permutations σ;
(ii) $[v_1, \ldots, v_{n+k}]$ if $v_1, \ldots, v_{n+k}$ do not span all of $\mathbb{Q}^n$; and
(iii) $[v_1, \ldots, v_{n+k}] - [av_1, v_2, \ldots, v_{n+k}]$ for all $a \in \mathbb{Q}^\times$.

The boundary map $\partial\colon Sh_k \to Sh_{k-1}$ is given by

$$\partial([v_1, \ldots, v_{n+k}]) = \sum_{i=1}^{n+k} (-1)^i [v_1, \ldots, \widehat{v_i}, \ldots v_{n+k}],$$

where as usual $\widehat{v_i}$ means to delete v_i.

The sharbly complex

$$\cdots \to Sh_i \to Sh_{i-1} \to \cdots \to Sh_1 \to Sh_0$$

is an exact sequence of $GL(n, \mathbb{Q})$-modules. We may define the Steinberg module St as the cokernel of $\partial\colon Sh_1 \to Sh_0$ (cf. [AGM12, Theorem 5]).

Of course, all these objects depend on n, which we suppress from the notation, since we will later only work with $n = 4$.

Let Γ be a congruence subgroup of $\mathrm{SL}(n, \mathbb{Z})$.

2.3. Definition. Let M be a right Γ-module, concentrated in degree 0. The *sharbly homology* of Γ with coefficients in M is defined to be $H_*(\Gamma, Sh_\bullet \otimes_\mathbb{Z} M)$, where Γ acts diagonally on the tensor product.

If (Γ, S) is a Hecke pair in $\mathrm{GL}(n, \mathbb{Z})$ and M is a right S-module, the Hecke algebra $\mathcal{H}(\Gamma, S)$ acts on the sharbly homology since S acts (diagonally) on $Sh_\bullet \otimes_{\mathbb{Z}} M$.

Here is a restatement of Corollary 8 of [AGM12], which shows the close connection between the sharbly homology and the group cohomology of Γ:

2.4. Theorem. *For any $\Gamma \subset GL(n, \mathbb{Z})$ and any coefficient module M in which 2 is invertible, there is a natural isomorphism of Hecke modules*

$$H_*(\Gamma, Sh_\bullet \otimes_{\mathbb{Z}} M) \to H_*(\Gamma, St \otimes_{\mathbb{Z}} M).$$

By Borel-Serre duality [BS73], if Γ is torsionfree, there is a natural isomorphism of Hecke modules

$$H_i(\Gamma, St \otimes_{\mathbb{Z}} M) \to H^{\binom{n}{2}-i}(\Gamma, M)$$

for all i. This result can be extended to any Γ as long as its torsion primes are invertible on M.

In general, the sharbly homology is more mysterious. Nevertheless, we still expect it to have number theoretic significance, as described in Conjecture 2.7 as follows.

2.5. Let $\Gamma_0(N)$ be the subgroup of matrices in $\mathrm{SL}(4, \mathbb{Z})$ whose first row is congruent to $(*, 0, 0, 0)$ modulo N. Define S_N to be the subsemigroup of integral matrices in $\mathrm{GL}(4, \mathbb{Q})$ satisfying the same congruence condition and having positive determinant relatively prime to $2N$.

Let $\mathcal{H}(N)$ denote the $\mathbb{Z}$-algebra of double cosets $\Gamma_0(N)S_N\Gamma_0(N)$. Then $\mathcal{H}(N)$ is a commutative algebra that acts on the cohomology and homology of $\Gamma_0(N)$ with coefficients in any $\mathbb{F}_2[S_N]$ module. When a double coset is acting on cohomology or homology, we call it a Hecke operator. Clearly, $\mathcal{H}(N)$ contains all double cosets of the form $\Gamma_0(N)D(\ell, k)\Gamma_0(N)$, where ℓ is a prime not dividing $2N$, $0 \leqslant k \leqslant m$, and

$$D(\ell, k) = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ell & & \\ & & & & \ddots & \\ & & & & & \ell \end{pmatrix}$$

is the diagonal matrix with the first $m - k$ diagonal entries equal to 1 and the last k diagonal entries equal to ℓ. It is known that these double cosets generate $\mathcal{H}(N)$ (cf. [Shi71, Thm. 3.20]). When we consider the double coset generated by $D(\ell, k)$ as a Hecke operator, we call it $T(\ell, k)$.

We can extend $\mathcal{H}(N)$ to a larger commutative algebra $\mathcal{H}^*(N)$ by adjoining the double cosets of $D(\ell, k)$ for $\ell \mid N$. Such a double coset, considered as a Hecke operator, is denoted $U(\ell, k)$.

Let $\overline{\mathbb{F}_2}$ be an algebraic closure of $\mathbb{F}_2$.

2.6. Definition. Let V be an $\mathcal{H}(N) \otimes_{\mathbb{Z}} \overline{\mathbb{F}_2}$-module. Suppose that $v \in V$ is a simultaneous eigenvector for all $T(\ell, k)$ and that $T(\ell, k)v = a(\ell, k)v$ with $a(\ell, k) \in \overline{\mathbb{F}_2}$ for all prime $\ell \nmid 2N$ and all $0 \leqslant k \leqslant 4$. If

$$\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}(4, \overline{\mathbb{F}_2})$$

is a continuous representation of $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ unramified outside $2N$, and

$$\sum_{k=0}^{4} (-1)^k \ell^{k(k-1)/2} a(\ell, k) X^k = \det(I - \rho(\mathrm{Frob}_\ell) X) \tag{1}$$

for all $\ell \not| 2N$, then we say that ρ is attached to v.

Here, Frob_ℓ refers to an arithmetic Frobenius element, so that if ε is the cyclotomic character, we have $\varepsilon(\mathrm{Frob}_\ell) = \ell$. The polynomial in (1) is called the *Hecke polynomial* for v and ℓ. If $\ell \mid N$, we can still compute the left-hand side of (1) and call it the Hecke polynomial for $U(\ell, k)$, but it has no obvious bearing on the attached Galois representation.

The following is a special case of [AGM11, Conjecture 5]:

2.7. Conjecture. *Let $N \geqslant 1$. Let v be a Hecke eigenclass in $H_*(\Gamma_0(N), Sh_\bullet \otimes_{\mathbb{Z}} \overline{\mathbb{F}_2})$. Then there is attached to v a continuous representation unramified outside $2N$,*

$$\rho\colon G_{\mathbb{Q}} \to \mathrm{GL}(4, \overline{\mathbb{F}_2}).$$

3. Computing homology and the Hecke action mod 2

3.1. As explained in Sections 5 and 6 of [AGM12], we compute the Hecke operators acting on sharbly cycles that are supported on Voronoi sharblies. Theorem 13 of [AGM12] guarantees that the packets of Hecke eignvalues we compute do occur on eigenclasses in $H_1(\Gamma_0(N), Sh_\bullet \otimes_{\mathbb{Z}} \overline{\mathbb{F}_2})$. In this section, we recall results from [AGM12] and explain how they are modified to work with $\mathbb{F}_2$ coefficients.

The sharbly complex is not finitely generated as a $\mathbb{Z}\mathrm{SL}(n, \mathbb{Z})$-module, which makes it difficult to use in practice to compute homology. To get a finite complex to compute H_1, we use the Voronoi complex. We refer to [AGM12, Section 5]) for any unexplained notation in what follows.

Let $X_n^0 \subset \mathbb{R}^{\binom{n+1}{2}}$ be the convex cone of positive-definite real quadratic forms in n-variables. This has a partial (Satake) compactification $(X_n^0)^*$ obtained by adjoining rational boundary components, which is itself a convex cone. The space $(X_n^0)^*$ can be partitioned into cones $\sigma = \sigma(x_1, \ldots, x_m)$, called *Voronoi cones*, where the x_i are contained in certain subsets of nonzero vectors from $\mathbb{Z}^n$. (We write elements of $\mathbb{Z}^n$ as row vectors, as we did in Section 2 for $\mathbb{Q}^n$.) The cones are built as follows: each nonzero $x_i \in \mathbb{Z}^n$ determines a rank 1 quadratic form $q(x_i) = {}^t x_i x_i \in (X_n^0)^*$. Let Π be the closed convex hull of the points $\{q(x) \mid x \in \mathbb{Z}^n, x \neq 0\}$. Then each of the proper faces of Π is a polytope, and the σs are exactly the cones on these polytopes. The indexing sets are constructed in the obvious way: if σ is the cone on $F \subset \Pi$, and F has distinct vertices $q(x_1), \ldots, q(x_m)$, then the indexing set is $\{\pm x_1, \ldots, \pm x_m\}$. We let Σ denote the set of all Voronoi cones.

Let X_n^* be the quotient of $(X_n^0)^*$ by homotheties. The images of the Voronoi cones are cells in X_n^*. Let $\mathbb{Z}V_\bullet$ be the oriented chain complex on these cells, graded by dimension, and let $\mathbb{Z}\partial V_\bullet$ be the subcomplex generated by those cells that do not meet the interior of X_n^* (i.e., the image in X_n^* of the positive-definite cone). The *Voronoi complex* is then defined to be $\mathcal{V}_\bullet = \mathbb{Z}V_\bullet / \mathbb{Z}\partial V_\bullet$. For our purposes, it is convenient to reindex $\mathcal{V}_\bullet$ by introducing the complex $\mathcal{W}_\bullet$, where $\mathcal{W}_k = \mathcal{V}_{n+k-1}$. The results of [AGM12] prove that if $n \leqslant 4$, both $\mathcal{W}_\bullet$ and $Sh_\bullet$ give resolutions of the Steinberg module. In particular, let $\Gamma = \Gamma_0(N)$. If M is a $\mathbb{Z}[\Gamma]$-module such that the the order of all torsion elements in Γ is invertible, then $H_*(\Gamma, \mathcal{W}_\bullet \otimes_{\mathbb{Z}} M) \approx$

$H_*(\Gamma, Sh_\bullet \otimes_{\mathbb{Z}} M)$, and furthermore by Borel–Serre duality are isomorphic (after reindexing) to $H^*(\Gamma, M)$. These two complexes can be related as follows in our case of interest: when $n = 4$, every Voronoi cell of dimension $\leqslant 5$ is a simplex. Thus for $0 \leqslant k \leqslant 2$, we can define a map of $\mathbb{Z}[\mathrm{SL}(4,\mathbb{Z})]$-modules

$$\theta_k \colon \mathcal{W}_k \to Sh_k$$

that takes the Voronoi cell $\sigma(v_1, \ldots, v_{k+4})$ to $\theta_k((v_1, \ldots, v_{k+4})) := [v_1, \ldots, v_{k+4}]$. This allows us to realize Voronoi cycles in these degrees in the sharbly complex.

In the current setting, in which $M \cong \overline{\mathbb{F}_2}$ with trivial Γ-action, all torsion orders in Γ are of course not invertible in M. Hence what we actually compute is more subtle. Let K denote either $\mathcal{W}_\bullet$ or $Sh_\bullet$. It is necessary to distinguish between $H_*(K) = H_*(\Gamma, K \otimes_{\mathbb{Z}} M)$, i.e. the homology of Γ with coefficients in the complex $K \otimes_{\mathbb{Z}} M$ and $H_1(K \otimes_\Gamma M)$, which is the homology of the complex $K \otimes_\Gamma M$ and which is the bottom line of a spectral sequence that computes $H_*(K)$. The Hecke algebra $\mathcal{H}$ acts on both of these homologies when $K = Sh_\bullet$, and the spectral sequence just mentioned is $\mathcal{H}$-equivariant.

Thus our computation begins by computing a basis $\{x_i\}$ of the homology group $H_1(\mathcal{W}_\bullet \otimes_\Gamma \mathbb{F}_2)$. We then compute elements $y_i = \theta_{1,*}(x_i) \in H_1(Sh_\bullet \otimes_\Gamma \mathbb{F}_2)$. Let T be a Hecke operator. We compute each Hecke translate Ty_i and then find a sharbly cycles z_i such that $z_i = Ty_i$ in $H_1(Sh_\bullet \otimes_\Gamma \overline{\mathbb{F}_2})$ and such that z_i is in the image of the map $\theta_{1,*}$. The inverse images $\theta_{1,*}^{-1}(z_i)$ can be written as linear combinations of the cycles x_i, which gives a matrix representing the action of T from which we can find eigenclasses and eigenvalues.

Unfortunately, as indicated in [AGM12, Section 6], we don't know if the map $\theta_{1,*}$ is injective. Thus this raises the question of what these eigenvalues mean. The answer is provided by Theorem 13 in [AGM12], which guarantees that if we find a cycle v representing a nonzero class in $H_1(\mathcal{W} \otimes_\Gamma \overline{\mathbb{F}}_2)$ such that $\theta_1(v)T$ is homologous to $a\theta_1(v)$ in $Sh_\bullet \otimes_\Gamma \overline{\mathbb{F}_2}$ (for a Hecke operator T), then there exists an eigenclass in $H_1(Sh_\bullet)$ with eigenvalue a for T. Hence eigenvalues we find in this way do occur in the sharbly homology and conjecturally are associated with Galois representations as in Conjecture 5 above.

3.2. Next we turn to the actual computation of the Hecke operators. Assume for the moment that Γ is torsionfree. Let $\xi = \sum n(x)x$ be a 1-sharbly cycle mod Γ, where all multiplicities $n(x)$ are taken to be nonzero. We also assume for the moment that 2 is invertible in the coefficient module. As described in [Gun00], we can encode ξ as a collection of 4-tuples $(x, n(x), \{y\}, \{L(y)\})$ of the following data:

1. The 1-sharbly x appears in $\xi = \sum n(x)x$ with multiplicity $n(x)$.
2. $\{y\}$ is the set of 0-sharblies appearing in the boundary of x.
3. For each 0-sharbly y in (2), the matrix $L(y)$ is a *lift* of y to $M_4(\mathbb{Z})$. In other words, the rows of the matrix $L(y)$ equal the entries of y, up to permutation and scaling by $\{\pm 1\}$.

We further require that the lift matrices in (3) are chosen Γ-equivariantly: suppose that for x, x' in the support of ξ there exist y (respectively y') appearing in the boundary of x (resp., x') with $y = y'\gamma$ for some $\gamma \in \Gamma$. Then we require $L(y) = L(y')\gamma$. Thus we have written ξ as a collection of 1-sharblies with multiplicities and with extra data that reflects the cycle structure of ξ mod Γ.

The congruence groups Γ we treat are not torsionfree in general, and we must modify the above data. When Γ has torsion, it can happen that a given 0-sharbly

y is taken to itself by a element of Γ that reverses orientation. In the language of [AGM02, Section 3.8], such Voronoi cells are *nonorientable*; in that paper and its sequels [AGM08, AGM10, AGM11] these cells are discarded when one computes $H_1(\mathcal{W})$. Unfortunately, these cells are not discardable when one computes Hecke operators using the ideas in [Gun00]: after applying a Hecke operator, such 0-sharblies must themselves be "reduced" to rewrite the Hecke translate in terms of cycles in the image of θ_1.

The point for the current discussion is that, when encoding ξ as a 4-tuple, any nonorientable 0-sharbly y must effectively have more than one lift matrix chosen for it. In particular, if y is nonorientable then we can find an orientation-reversing γ in the stabilizer of y with $y\gamma^2 = y$, and we must replace the tuple $\Phi = (x, n(x), \{y\}, \{L(y)\})$ in our data with a *pair* of tuples Φ', Φ''. These tuples are the same as Φ except that (i) if Φ has multiplicity $n(x)$, then Φ', Φ'' each have multiplicity $n(x)/2$, and (ii) if Φ' has a lift matrix $L(y)$ for y, then Φ'' has the lift matrix $L(y)\gamma$ for y in the same position.

Hence we "split" the contribution $n(x)x$ to ξ into a contribution of two 1-sharblies, each of multiplicity $n(x)/2$, so that we can encode it as two 4-tuples that can maintain the Γ-equivariance of the data. Of course there may be more nonorientable 0-sharblies in the boundary of x than just y. If so we continue to split tuples as needed, dividing multiplicities by 2 along the way. Since x has at most 5 0-sharblies in its boundary, our original x gives rise to at most 2^5 tuples.

We now return to the case at hand, in which 2 is not invertible in the coefficients. Clearly we cannot apply the above construction to encode ξ as a collection of tuples, since we cannot replace $n(x)$ by $n(x)/2$ if a 0-sharbly is taken to itself by its stabilizer. Fortunately we are saved by an observation of Dan Yasaki: since $-1 = 1$ in the coefficients, there is no distinction between orientable and nonorientable Voronoi cells! All Voronoi cells are orientable; none are discarded when one builds the complex $\mathcal{W}_\bullet$. The consequence is that a sharbly chain *never* becomes a cycle mod Γ because of orientation-reversing self-maps on 0-sharbles in its boundary. Hence we never have to divide by 2 in building the tuples Φ to encode ξ.

4. Finding attached Galois representations

4.1. Suppose we have a finite-dimensional $\mathbb{F}_2$-vector space V with a Hecke action. We now describe how we find Galois representations that are conjecturally attached to Hecke eigenvectors in $V \otimes_{\mathbb{F}_2} \overline{\mathbb{F}_2}$. Our Galois representation finder is a Python script built on the mathematical software package Sage [S$^+$12].

4.2. We start by using the algorithm in [Gun00] to compute explicitly the Hecke operators $T(\ell, k)$ for $k = 1, 2, 3$ and for ℓ ranging through a set L of small odd primes. The operator is $U(\ell, k)$ rather than $T(\ell, k)$ if $\ell \mid N$. The L we use depends on N as in Table 1. We use a larger L when N is smaller, because the computations are faster for smaller N.

Let $\mathbb{F}$ be the field generated over $\mathbb{F}_2$ by the eigenvalues of the Hecke operators we have computed. $\mathbb{F}$ is a finite extension of $\mathbb{F}_2$. We replace V with its extension of scalars $V \otimes_{\mathbb{F}_2} \mathbb{F}$ for the rest of the discussion.

For each operator we have computed, we decompose V into eigenspaces under that operator. Then we take the common refinement of all the decompositions. In other words, let E have the form $\bigcap_{(\ell,k)} E_{\ell,k}$, where $E_{\ell,k}$ is any one of the eigenspaces for the operator at (ℓ, k), and the intersection is over all $\ell \in L$ and $k = 1, 2, 3$. We

Table 1. We compute $T(\ell, k)$ and $U(\ell, k)$ at level N for the ℓ shown in L.

N	L
3–10, 17	$\{3, 5, 7, 11, 13\}$
11	$\{3, 5, 7, 11, 13, 17\}$
13	$\{3, 5, 7, 11\}$
other	$\{3, 5, 7\}$

find all the non-zero E of this form, and call each a *simultaneous eigenspace*. The E's are pairwise disjoint, and together they span a subspace of V. By construction, the Hecke eigenvalues $a(\ell, k)$ are constant on each E and characterize it. The function $(\ell, k) \mapsto a(\ell, k)$ is the *Hecke eigenpacket* of E.

To a simultaneous eigenspace E we now attach a family of polynomials. Let $L' = \{\ell \in L \mid \ell \nmid N\}$.

4.3. Definition. The *polynomial system* $\mathcal{F}(E)$ is the mapping that sends $\ell \in L'$ to the Hecke polynomial with eigenvalues $a(\ell, k)$ defined in (1).

The Hecke polynomials have coefficients in the field of eigenvalues $\mathbb{F}$, but they do not necessarily split into linear factors over that field. We enlarge $\mathbb{F}$ if necessary so that all the Hecke polynomials for $\ell \in L'$ split into linear factors in $\mathbb{F}[X]$, and again we replace V with its extension of scalars $V \otimes_{\mathbb{F}_2} \mathbb{F}$. The largest $\mathbb{F}$ we have had to work with is $\mathbb{F}_{64}$ at level $N = 59$, a very small field from the computational standpoint.

4.4. We will be using various Galois representations ρ that have been defined classically. Each ρ is a continuous, semisimple representation of $G_{\mathbb{Q}}$ unramified outside $2N$. It takes values in $\mathrm{GL}(n', \mathbb{F})$ for $n' = 1$ or 2, where $\mathbb{F}$ is the particular finite extension of $\mathbb{F}_2$ described above. The characteristic polynomial of Frobenius for ρ is known and is of degree n' for each $\ell \nmid 2N$.

4.5. Definition. The *polynomial system* $\mathcal{F}(\rho)$ is the mapping that sends $\ell \in L'$ to the characteristic polynomial of Frobenius for ρ at ℓ.

Before we say which ρ we consider, let us describe how we conjecturally attach a sum of ρ's to a simultaneous eigenspace E. Say that $\mathcal{F}(\rho)$ *divides* $\mathcal{F}(E)$ if, for each $\ell \in L'$, the polynomial at ℓ for ρ divides the polynomial at ℓ for E. When one polynomial system divides another, define the *quotient system* in the obvious way.

For a given E, let $\mathcal{F} = \mathcal{F}(E)$ be its polynomial system. We run through a list of Galois representations ρ in some fixed order. The first time we find a ρ (call it ρ_1) whose system divides $\mathcal{F}$, we replace $\mathcal{F}$ by the quotient system. If the system for ρ_1 divides $\mathcal{F}$ more than once (say n_1 times), we take the quotient n_1 times. After that, we continue running through the rest of the ρ's in our fixed order. When we find a ρ_2 whose system divides the new $\mathcal{F}$, say n_2 times, we again replace $\mathcal{F}$ with the quotient system. We stop with success when $\mathcal{F}$ becomes the trivial system, meaning all polynomials have degree zero. We stop with failure when we run out of ρ's before $\mathcal{F}$ becomes trivial. In the successful cases, we say that the *Galois representation apparently attached to* E is

$$\rho_1^{\oplus n_1} \oplus \rho_2^{\oplus n_2} \oplus \cdots .$$

The word "apparently" means that this Galois representation matches our Hecke data as far as our data extends.

4.6. Now we describe the Galois representations ρ we use. We have two different lists of Galois representations, $\boldsymbol{\rho}_{2,4}$ and $\boldsymbol{\rho}_{2,3}$. With either list, we always successfully find a Galois representation that is apparently attached to one of our simultaneous eigenspaces E. Specific results are in Section 5. The lists $\boldsymbol{\rho}_{2,4}$ and $\boldsymbol{\rho}_{2,3}$ are ordered, and the order matters in the following sense. When we split the first representation, ρ_1, off of E, we want ρ_1 to be as simple as possible. ρ_2 should be the second simplest, and so on.

In this subsection, we define $\boldsymbol{\rho}_{2,4}$ and $\boldsymbol{\rho}_{2,3}$. In (4.7)–(4.9), we give the motivation behind the definitions.

$\boldsymbol{\rho}_{2,4}$ begins with a list of one-dimensional Galois representations χ. These are Dirichlet characters with value in $\mathbb{F}$, which we identify with one-dimensional representations as usual. Let M be the odd part of N. The definition is that a Dirichlet character χ belongs to $\boldsymbol{\rho}_{2,4}$ if and only if the conductor N_1 of χ is a divisor of M. Following the intuition that a Dirichlet character with smaller conductor is simpler than one with a larger conductor, we put the Dirichlet characters into $\boldsymbol{\rho}_{2,4}$ in order of increasing N_1. For instance, $\chi = 1$ comes first. Sage's class `DirichletGroup` enumerates the χ for a given N_1 automatically. The characteristic polynomial of Frobenius at ℓ for χ is $1 + \chi(\ell)X$, for all $\ell \nmid 2N$.

After the Dirichlet characters, we put into $\boldsymbol{\rho}_{2,4}$ certain Galois representations ρ coming from classical cusp forms for congruence subgroups of $\mathrm{SL}(2,\mathbb{Z})$. We emphasize that the cusp forms are in characteristic zero, though the ρ take values in characteristic two. The characteristic polynomials of Frobenius for the cusp forms are naturally defined over number fields, so, as we describe which cusp forms we use, we must also describe how we reduce to get Galois representations defined over $\mathbb{F}$.

Let N_1 be a divisor of M. Let f be a newform of weight 2 or 4 for $\Gamma_0(N_1)$. The coefficients of the q-expansion of f generate a number field K_f, with ring of integers $\mathcal{O}_{K_f}$. Let $\mathfrak{p}$ be a prime of K_f over 2. If $\mathbb{F}$ is of high enough degree over $\mathbb{F}_2$, then the finite field $\mathcal{O}_{K_f}/\mathfrak{p}$ has an embedding $\alpha_\mathfrak{p}$ into $\mathbb{F}$. In every case we have computed, $\mathbb{F}$ is indeed large enough so that this embedding exists. Then the pair $(f,\mathfrak{p})$ gives rise to a Galois representation ρ into $\mathrm{GL}(2,\mathbb{F})$, by reduction mod $\mathfrak{p}$ composed with $\alpha_\mathfrak{p}$. For any $\ell \nmid 2N$, the characteristic polynomial of Frobenius is $1 - \alpha_\mathfrak{p}(a_\ell)X + X^2$, where a_ℓ is the ℓ-th coefficient in the q-expansion of f.

By definition, $\boldsymbol{\rho}_{2,4}$ contains the representation ρ for $(f,\mathfrak{p})$, for all $N_1 \mid M$ and all newforms f of weight 2 or 4 for $\Gamma_0(N_1)$. The order is as follows. The outermost loop is over weight 2 first, then weight 4. For a given weight, we let N_1 run through the divisors of M in increasing order. We find the newforms f for $\Gamma_0(N_1)$ and the given weight. Sage's class `CuspForms`, with its method `newforms`, makes this last step automatic. For each newform f, we find the number field K_f. If there is more than one f for the given weight and N_1, we sort these f's by two keys; the primary key says the degree $[K_f : \mathbb{Q}]$ should be increasing, and the secondary key says that the absolute value of the discriminant of K_f should be increasing.

We now turn to the definition of $\boldsymbol{\rho}_{2,3}$. It begins with the same Dirichlet characters as $\boldsymbol{\rho}_{2,4}$, in the same order. Next, let $N_1 \mid M$. Let ψ be a character on $\mathbb{Z}/N_1\mathbb{Z}$. Let f be a newform of weight 2 or 3 with level N_1 and nebentype character ψ. Let

K_f and $\mathfrak{p}$ be as before. The pair $(f, \mathfrak{p})$ gives rise to a Galois representation ρ into $\mathrm{GL}(2, \mathbb{F})$ as above.

By definition, $\boldsymbol{\rho}_{2,3}$ contains the representation ρ for $(f, \mathfrak{p})$, for all $N_1 \mid M$, all ψ, and all newforms f of weight 2 or 3 of level N_1 and nebentype character ψ. The order is as follows. The outermost loop is over weight 2 first, then weight 3. For a given weight, we let N_1 run through the divisors of M in increasing order. For a given N_1, we run through the ψ in the order Sage uses, which is to fix generators of the character group and raise them to powers in lexicographic order, starting with 0-th powers. In particular, the trivial ψ comes first. We find the newforms for the given weight and ψ, again using Sage's class `CuspForms`. For each newform f, we find the number field K_f, and sort the f's by degree and discriminant as before.

4.7. The definitions of $\boldsymbol{\rho}_{2,4}$ and $\boldsymbol{\rho}_{2,3}$ present two different perspectives on how Galois representations would be attached to our homology classes.

Our construction of $\boldsymbol{\rho}_{2,4}$ reflects a guess based on an analogy with our first papers in this series, which studied homology in characteristic zero [AGM02, AGM08, AGM10]. In them, we found that, for small levels, all the homology appeared to be accounted for by classes supported on the Borel-Serre boundary, and that it was always related to Dirichlet characters and classical cuspforms of weights 2 and 4. Although with this guess we might expect to need newforms of even level dividing N, in practice we did not.

By contrast, the list $\boldsymbol{\rho}_{2,3}$ reflects the conjecture found in [ADP02]. Here we seek mod 2 Galois representations which the conjecture would associate to a homology class of level N, weight k and trivial nebentype. In particular, the Serre conductor of such Galois representations would divide the odd part M of the level N. We never get to test this for more than two-dimensional Galois representations, not all the way to four dimensions, because we keep splitting off the Dirichlet characters. Our guesses for two-dimensional representations are that they are mod 2 Galois representations which Serre's conjecture would attach to a homology class of level M, weight k and trivial nebentype. But we don't have a good way to construct these mod 2 objects, except by reducing characteristic-zero modular forms mod 2. Kevin Buzzard tells us that we are guaranteed to find all such two-dimensional mod 2 Galois representations by looking at modular forms of level M, weights 2 or 3, and a range of nebentypes. In (4.8) let us explain this guarantee.

4.8. We thank Kevin Buzzard for much of the information in this subsection. Let σ be a mod 2 Galois representation which Serre's conjecture (now a theorem of Khare-Wintenberger [KW09a, KW09b]) would attach to a homology class of level M, weight k and trivial nebentype. The following arguments are valid for all p prime to M, including $p = 2$ [Edi92, Thm. 4.3]. First of all, we do not have to worry about $k = 1$, because by multiplying by the Hasse invariant we can move to $k = p = 2$. Thus we may assume $k \geqslant 2$. Any eigenform will show up, up to a twist, in weight at most $p+1$. Thus for $p = 2$, where there are no twists at all, we need only compute in weights $k = 2$ and 3. All the mod 2 eigenforms lift to characteristic zero, because $k \geqslant 2$. Because $p \leqslant 3$, we cannot guarantee that the nebentype character lifts to the character we expect; however, we know that it lifts to *some* character. Since the eigenforms lift to characteristic zero, well-known work of Deligne attaches p-adic representations to them. In turn, these reduce to mod p representations, one of which is the given σ.

In practice, there are a large number of nebentypes ψ, and often a large number of cusp forms for a given nebentype. We cut down on the amount of computation as follows. Our desired four-dimensional Galois representation must have determinant 1. In every case where we use a cusp form, we have already split off two Dirichlet characters; that is, the four-dimensional representation is $\chi_1 \oplus \chi_2 \oplus \rho_3$ where χ_1, χ_2 are Dirichlet characters and ρ_3 is from a cusp form of nebentype ψ. Thus $\det \rho_3 = \delta$, where we define $\delta = \det(\chi_1\chi_2)^{-1}$. Furthermore, $\det \rho_3 = \det \psi$. When we construct the list $\boldsymbol{\rho}_{2,3}$ is our program, we already know χ_1 and χ_2, so we make the list smaller by only including ψ that are congruent to δ mod 2.

4.9. Let Δ be the group of characters $\psi : (\mathbb{Z}/M\mathbb{Z})^\times \to \mathbb{C}$ that are congruent to 1 mod 2. The ψ we need to use are the coset of Δ translated by δ, so we would like to understand Δ. Let μ be the exponent of the group $(\mathbb{Z}/M\mathbb{Z})^\times$. All our ψ take values in $\mathbb{Q}(\zeta_\mu)$, the cyclotomic field of μ-th roots of unity, and "mod 2" means modulo a prime ideal $\mathfrak{p}_\mu$ over 2 in $\mathbb{Q}(\zeta_\mu)$. Let ν be the power of 2 dividing μ, and let o be the odd part, so that $\mu = \nu o$. As usual, $\mathbb{Q}(\zeta_\mu)$ is the compositum of $\mathbb{Q}(\zeta_\nu)$ and $\mathbb{Q}(\zeta_o)$, and $\mathfrak{p}_\mu$ can be understood by studying the primes $\mathfrak{p}_\nu$, $\mathfrak{p}_o$ over 2 in their respective fields.

4.10. Lemma. *Δ is the group of characters whose image lies in $\mathbb{Q}(\zeta_\nu)$. Equivalently, it is the group of characters whose orders are pure powers of two dividing ν.*

To prove the lemma, first consider ν. In $\mathbb{Q}(\zeta_\nu)$, 2 is totally ramified, and $\mathfrak{p}_\nu = (2, 1-\zeta_\nu)$ is the only prime over 2. Any Dirichlet character is 1 mod 2, because ζ_ν and all its powers are congruent to 1 mod $\mathfrak{p}_\nu$. Second, consider o. For an odd prime q that divides o, let o' be maximal power of q that divides o. In $\mathbb{Q}(\zeta_{o'})$, 2 is unramified. Under the mapping to the residue class field, the o' distinct powers of $\zeta_{o'}$ map to o' distinct values, so only the trivial power $\zeta_{o'}^0 = 1$ maps to 1 mod 2. That is, only the trivial Dirichlet character is 1 mod $\mathfrak{p}_{o'}$. The lemma follows from the Chinese remainder theorem. □

5. Results

For the list $\boldsymbol{\rho}_{2,4}$, subsection (5.1) contains a table of results for several levels N. For each level N, we first give the overall dimension of the H_1 we compute. Each succeeding row describes a simultaneous eigenspace E. The first two columns in the row give the type of E, a Roman numeral to be defined below, followed by $\dim E$.

Let $\mathbf{1}$ be the trivial one-dimensional Galois representation. Roman numeral I means that the Galois representation apparently attached to our Hecke eigenspace is the sum of four trivial representations, $\mathbf{1}^{\oplus 4}$. The symbol I_m means the representation is the sum of two trivial and two non-trivial representations, $\mathbf{1} \oplus \mathbf{1} \oplus \chi_m \oplus \bar{\chi}_m$. The non-trivial representations go to $\mathbb{F}_4$ rather than $\mathbb{F}_2$. More precisely, χ_m maps $(\mathbb{Z}/m\mathbb{Z})^\times$ surjectively to $\mathbb{F}_4^\times$, and $\bar{\chi}_m$ is its conjugate under $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. These statements characterize χ_m and $\bar{\chi}_m$ up to conjugation.

Roman numerals II and IV mean that the Galois representation apparently attached to our Hecke eigenspace is the sum of two $\mathbf{1}$'s and the Galois representation attached to a cuspidal newform from $\boldsymbol{\rho}_{2,4}$. The newform has weight 2 or 4, respectively. The congruence subgroup is $\Gamma_0(N)$, where N is the level where the representation first appears in the tables.

In subsection (5.2), we present data for $\boldsymbol{\rho}_{2,3}$, but list only the representations where $\boldsymbol{\rho}_{2,4}$ and $\boldsymbol{\rho}_{2,3}$ give different results. Roman numeral III stands for the sum of two $\mathbf{1}$'s and a cuspidal newform of weight 3 from $\boldsymbol{\rho}_{2,3}$.

Our tables do not give the Hecke polynomials of the $T(\ell, k)$. This is because the Hecke polynomials can easily be recovered from the Galois representation. For example, all the T's for a type I representation have Hecke polynomial $(x+1)^4$. The Hecke polynomials for the $U(\ell, k)$ are described below. The list of ℓ's used for a given N was given in Table 1.

For types II, III, and IV, we give details about the cusp form in the third column of each row. The coefficients of the q-expansions are in $\mathbb{Q}$ unless the number field is indicated. We write $i = \sqrt{-1}$ as usual.

We observe that the results for types I, I_m, and II are always the same at a given level for $\boldsymbol{\rho}_{2,4}$ and $\boldsymbol{\rho}_{2,3}$. The only differences we see are when type IV changes to type III. It is somewhat surprising that the type II representations never change between $\boldsymbol{\rho}_{2,4}$ and $\boldsymbol{\rho}_{2,3}$. For $\boldsymbol{\rho}_{2,4}$ and weight 2, we always searched for cusp forms on $\Gamma_0(N_1)$, which means $\Gamma_1(N_1)$ with trivial nebentype. For $\boldsymbol{\rho}_{2,3}$ and weight 2, we searched for cusp forms of all nebentypes. The observation is that our program produced a weight 2 cusp form for some nebentype if and only if that nebentype was trivial.

The same cusp form can appear at the same level N for different simultaneous eigenspaces. This reflects the different embeddings of the number fields into $\mathbb{F}$. For example, in the last table of (5.1), with $\boldsymbol{\rho}_{2,4}$ and $N = 59$, the same weight-2 cusp form appears four times, in all four of the type II representations. The cusp form is defined over a quintic extension of $\mathbb{Q}$. We find that 2 factors in the quintic field as a product $\mathfrak{p}_1\mathfrak{p}_2$ of prime ideals, where $\mathfrak{p}_1$ is unramified and has residue class field $\mathbb{F}_8$, while $\mathfrak{p}_2$ has ramification index 2 and residue class field $\mathbb{F}_2$. The first three occurrences of the cusp form belong to $\mathfrak{p}_1$. Let ϖ be a root of $x^3 + x + 1 = 0$ in $\mathbb{F}_8$. The Hecke polynomials for the first representation are

$$
\begin{aligned}
(x+1)^2 \cdot (x^2 + \varpi^2 x + 1) & \qquad (\ell = 3) \\
(x+1)^2 \cdot (x^2 + \varpi x + 1) & \qquad (\ell = 5) \\
(x+1)^2 \cdot (x^2 + (\varpi^2 + \varpi) x + 1) & \qquad (\ell = 7)
\end{aligned}
$$

The Galois group $\mathrm{Gal}(\mathbb{F}_8/\mathbb{F}_2)$ permutes ϖ, ϖ^2, and $\varpi^4 = \varpi + \varpi^2$ in a three-cycle. Checking the Hecke polynomials of the second and third Galois representations, we see that these three representations (those with eigenspaces of dimension 4) are permuted in a three-cycle by the Galois group. The fourth occurrence of the cusp form (dimension 15) is for $\mathfrak{p}_2$; here the coefficients of the Hecke polynomials are down in $\mathbb{F}_2$.

For a given N, the sum of the dimensions of the simultaneous eigenspaces is often less than the dimension of the full H_1. This is because many Hecke operators, both $T(\ell, k)$ and $U(\ell, k)$, turn out not to be semisimple.

Every level N we have computed has some representations of type I. A few have type I_m. To avoid cluttering the tables, we list these representations here. The notation (N, d) means level N has a representation with corresponding eigenspace of dimension d. When the same N occurs in more than one pair, there are simultaneous eigenspaces where the $T(\ell, k)$ act the same but the $U(\ell, k)$ act differently.

- The type I representations that appear to be attached to our data are $(3, 1)$, $(4, 1)$, $(5, 1)$, $(6, 5)$, $(7, 3)$, (8.6), $(9, 1)$, $(9, 4)$, $(10, 7)$, $(11, 1)$, $(12, 19)$,

$(13,1)$, $(14,13)$, $(15,14)$, $(16,17)$, $(17,6)$, $(18,5)$, $(18,16)$, $(19,1)$, $(20,30)$, $(21,16)$, $(22,5)$, $(23,3)$, $(24,55)$, $(25,1)$, $(25,9)$, $(26,7)$, $(27,1)$, $(27,3)$, $(27,4)$, $(28,43)$, $(29,1)$, $(30,59)$, $(31,3)$, $(32,40)$, $(33,14)$, $(34,29)$, $(35,18)$, $(36,19)$, $(36,50)$, $(37,1)$, $(38,5)$, $(39,21)$, $(41,8)$, $(43,10)$, $(47,3)$, $(53,1)$, $(59,1)$.

- We find one type I_9 representation of dimension 2 at level 27.
- We find one type I_7 representation of dimension 4 at level 35.

We use the operators $U(\ell,k)$ to divide up the simultaneous eigenspaces as finely as possible, and we compute their Hecke polynomials, but we do not consider the $U(\ell,k)$ when attaching Galois representations. Again, to avoid cluttering the tables with $U(\ell,k)$ data, we summarize their Hecke polynomials here. The general rule is that, when ℓ is an odd prime dividing N, the Hecke polynomial of U is $x^4+x^3+x^2+x+1$. We list the exceptions in the format (N, U_ℓ, d), which means that for all the representations with a d-dimensional eigenspace we have found at level N, the operator $U(\ell,k)$ has the Hecke polynomial described.

- The Hecke polynomial is $(x^2+x+1)^2$ for (9, U_3, 4), (18, U_3, 16), (25, U_5, 2 or 9), (27, U_3, 2 or 4) and (36, U_3, 50).
- The Hecke polynomial is x^4+x^3+1 for (27, U_3, 3).
- Let ω be a primitive cube root of unity in $\mathbb{F}_4$. At level $N=33$, the Hecke polynomial for U_3 is $x^4+\omega x^3+x^2+\omega x+1$ on one of the 4-dimensional eigenspaces; for the other 4-dimensional eigenspace, it is the polynomial's conjugate under $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$, namely $x^4+(\omega+1)x^3+x^2+(\omega+1)x+1$. At level $N=39$, the same pair of conjugate Hecke polynomials occur for U_3 and the pair of 2-dimensional eigenspace, for both types III and IV.
- The Hecke polynomial is x^4+x+1 for (33, U_3, 9), and also for (39, U_3, 4) for both types III and IV.

5.1. Here are the results of types II and IV for $\boldsymbol{\rho}_{2,4}$. (Types I and I_m were described above.)

Level 11. Dimension 5.		
II	4	$\rho_{2,11}=q-2q^2-q^3+2q^4+q^5+O(q^6)$
Level 13. Dimension 5.		
IV	2	$\rho_{4,13}=q-5q^2-7q^3+17q^4-7q^5+O(q^6)$
Level 19. Dimension 9.		
II	4	$\rho_{2,19}=q-2q^3-2q^4+3q^5+O(q^6)$
IV	2	$\rho_{4,19}=q-3q^2-5q^3+q^4-12q^5+O(q^6)$
Level 23. Dimension 12.		
II	9	$q+b_0q^2+(-2b_0-1)\,q^3+(-b_0-1)\,q^4+2b_0q^5+O(q^6)$, with $b_0=(-1+\sqrt{5})/2$.
Level 25. Dimension 14.		
IV	2	$q-q^2-7q^3-7q^4+O(q^6)$
Level 26. Dimension 25.		
IV	10	$\rho_{4,13}$
Level 27. Dimension 20.		
II	4	$q-2q^4+O(q^6)$
Level 29. Dimension 17.		
II	5	$q+b_0q^2-b_0q^3+(-2b_0-1)\,q^4-q^5+O(q^6)$, with $b_0=-1+\sqrt{2}$.

Level 31. Dimension 16.		
II	9	$q + b_0q^2 - 2b_0q^3 + (b_0 - 1)\,q^4 + q^5 + O(q^6)$, with $b_0 = (1 + \sqrt{5})/2$.
Level 33. Dimension 35.		
II	4	$\rho_{2,11}$
II	4	$\rho_{2,11}$
II	9	$\rho_{2,11}$
Level 37. Dimension 21.		
II	12	$q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + O(q^6)$
IV	2	$\rho_{4,37} = q + b_0q^2 + \left(-\frac{1}{8}b_0^3 - \frac{9}{8}b_0^2 - \frac{13}{4}b_0 - \frac{11}{4}\right) q^3 + \left(b_0^2 - 8\right) q^4 + \left(\frac{13}{8}b_0^3 + \frac{85}{8}b_0^2 + \frac{25}{4}b_0 - \frac{93}{4}\right) q^5 + O(q^6)$, with $b_0^4 + 6b_0^3 - b_0^2 - 16b_0 + 6 = 0$.
IV	2	$\rho_{4,37}$
Level 38. Dimension 40.		
II	15	$\rho_{2,19}$
IV	10	$\rho_{4,19}$
Level 39. Dimension 41.		
IV	2	$\rho_{4,13}$
IV	2	$\rho_{4,13}$
IV	4	$\rho_{4,13}$
Level 43. Dimension 26.		
IV	2	$\rho_{4,43} = q + b_0q^2 + \left(\frac{1}{8}b_0^3 + \frac{1}{8}b_0^2 - \frac{7}{2}b_0 - \frac{13}{4}\right) q^3 + \left(b_0^2 - 8\right) q^4 + \left(-\frac{9}{8}b_0^3 - \frac{33}{8}b_0^2 + \frac{19}{2}b_0 + \frac{5}{4}\right) q^5 + O(q^6)$, with $b_0^4 + 4b_0^3 - 9b_0^2 - 14b_0 + 2 = 0$.
IV	2	$\rho_{4,43}$
Level 47. Dimension 25.		
II	9	$\rho_{2,47} = q + b_0q^2 + \left(b_0^3 - b_0^2 - 6b_0 + 4\right) q^3 + \left(b_0^2 - 2\right) q^4 + \left(-4b_0^3 + 2b_0^2 + 20b_0 - 10\right) q^5 + O(q^6)$, with $b_0^4 - b_0^3 - 5b_0^2 + 5b_0 - 1 = 0$.
II	9	$\rho_{2,47}$
Level 53. Dimension 33.		
II	8	$q - q^2 - 3q^3 - q^4 + O(q^6)$
IV	2	$q + b_1q^2 + \left(\frac{1}{14}b_1^3 - \frac{3}{14}b_1^2 - \frac{37}{14}b_1 - \frac{3}{2}\right) q^3 + \left(b_1^2 - 8\right) q^4 + \left(-\frac{5}{14}b_1^3 - \frac{13}{14}b_1^2 + \frac{31}{14}b_1 + \frac{3}{2}\right) q^5 + O(q^6)$, with $b_1^4 + 4b_1^3 - 16b_1^2 - 42b_1 + 49 = 0$.
Level 59. Dimension 36.		
II	4	$\rho_{2,59} = q + b_0q^2 + \left(-\frac{1}{4}b_0^4 + \frac{5}{4}b_0^2 - \frac{1}{2}b_0\right) q^3 + \left(b_0^2 - 2\right) q^4 + \left(\frac{3}{4}b_0^4 + \frac{1}{2}b_0^3 - \frac{23}{4}b_0^2 - 3b_0 + 7\right) q^5 + O(q^6)$, with $b_0^5 - 9b_0^3 + 2b_0^2 + 16b_0 - 8 = 0$.
II	4	$\rho_{2,59}$
II	4	$\rho_{2,59}$
II	15	$\rho_{2,59}$
IV	4	$q + b_1q^2 + (-3b_1 + 1)\,q^3 + (b_1 - 4)\,q^4 + (3b_1 - 17)\,q^5 + O(q^6)$, with $b_1 = (1 + \sqrt{17})/2$.

5.2. Here are the results for $\boldsymbol{\rho}_{2,3}$, where they differ from $\boldsymbol{\rho}_{2,4}$.

Level 13. Dimension 5.		
III	2	$\rho_{3,13} = q + b_0q^2 + ((i - 1)\,b_0 - 3)\,q^3 + ((-2i - 2)\,b_0 - i)\,q^4 + (b_0 + 3i + 3)\,q^5 + O(q^6)$, for $\Gamma_1(13)$ with nebentype mod 13 mapping $2 \mapsto i$, with coefficients in $\mathbb{Q}(i)[b_0]/(b_0^2 + (2i + 2)\,b_0 - 3i)$.

Level 19. Dimension 9.		
III	2	$\rho_{3,19} = q + b_1q^2 - b_1q^3 - 9q^4 + 4q^5 + O(q^6)$, for $\Gamma_1(19)$ with nebentype mod 19 mapping $2 \mapsto -1$, where $b_1 = \sqrt{-13}$.
Level 25. Dimension 14.		
III	2	$q + b_0q^2 + ib_0q^3 - iq^4 + O(q^6)$, for $\Gamma_1(25)$ with nebentype mod 25 mapping $2 \mapsto i$, with coefficients in $\mathbb{Q}(i)[b_0]/(b_0^2 - 3i)$.
Level 26. Dimension 25.		
III	10	$\rho_{3,13}$
Level 37. Dimension 21.		
III	2	$\rho_{3,37} = q + b_0q^2 + \left(\frac{1}{4}ib_0^4 + \left(\frac{1}{4}i - \frac{1}{4}\right)b_0^3 + \frac{11}{4}b_0^2 + \left(\frac{5}{4}i + \frac{5}{4}\right)b_0 - 3i\right)q^3 + \left(b_0^2 - 4i\right)q^4 + \left(-\frac{1}{4}ib_0^5 + \left(-\frac{1}{2}i + \frac{1}{2}\right)b_0^4 - \frac{13}{4}b_0^3 + (-5i - 5)\,b_0^2 + \frac{17}{2}ib_0 + 6i - 6\right)q^5 + O(q^6)$, for $\Gamma_1(37)$ with nebentype mod 37 mapping $2 \mapsto i$, with coefficients in $\mathbb{Q}(i)[b_0]/(b_0^6 + (3i+3)\,b_0^5 - 10ib_0^4 + (-34i + 34)\,b_0^3 - 5b_0^2 + (-59i - 59)\,b_0 - 24i)$.
III	2	$\rho_{3,37}$
Level 38. Dimension 40.		
III	10	$\rho_{3,19}$
Level 39. Dimension 41.		
III	2	$\rho_{3,39} = q + b_0q^2 + ((i-1)\,b_0 - 3)\,q^3 + ((-2i-2)\,b_0 - i)\,q^4 + (b_0 + 3i + 3)\,q^5 + O(q^6)$, for $\Gamma_1(13)$ with nebentype mod 13 mapping $2 \mapsto i$, with coefficients in $\mathbb{Q}(i)[b_0]/(b_0^2 + (2i+2)\,b_0 - 3i)$.
III	2	$\rho_{3,39}$
III	4	$\rho_{3,39}$
Level 43. Dimension 26.		
III	2	$\rho_{3,43} = q + b_1q^2 + \left(-\frac{1}{4}b_1^5 - \frac{15}{4}b_1^3 - \frac{25}{2}b_1\right)q^3 + \left(b_1^2 + 4\right)q^4 + \left(\frac{1}{4}b_1^5 + \frac{11}{4}b_1^3 + \frac{9}{2}b_1\right)q^5 + O(q^6)$, for $\Gamma_1(43)$ with nebentype mod 43 mapping $3 \mapsto -1$, with coefficients in $\mathbb{Q}[b_1]/(b_1^6 + 20b_1^4 + 121b_1^2 + 214)$.
III	2	$\rho_{3,43}$
Level 53. Dimension 33.		
III	2	$q + b_0q^2 + \left(-\frac{39}{578}ib_0^7 + \left(-\frac{91}{578}i + \frac{91}{578}\right)b_0^6 - \frac{649}{578}b_0^5 + \left(-\frac{1499}{578}i - \frac{1499}{578}\right)b_0^4 + \frac{2233}{578}ib_0^3 + \left(\frac{2666}{289}i - \frac{2666}{289}\right)b_0^2 + \frac{219}{578}b_0 + \frac{861}{289}i + \frac{861}{289}\right)q^3 + \left(b_0^2 - 4i\right)q^4 + \left(-\frac{15}{578}b_0^7 + \left(-\frac{35}{578}i - \frac{35}{578}\right)b_0^6 + \frac{383}{578}ib_0^5 + \left(\frac{621}{578}i - \frac{621}{578}\right)b_0^4 + \frac{2993}{578}b_0^3 + \left(\frac{1181}{289}i + \frac{1181}{289}\right)b_0^2 - \frac{6131}{578}ib_0 - \frac{220}{289}i + \frac{220}{289}\right)q^5 + O(q^6)$, for $\Gamma_1(53)$ with nebentype mod 53 mapping $2 \mapsto i$, with coefficients in $\mathbb{Q}(i)[b_0]/(b_0^8 + (3i+3)\,b_0^7 - 16ib_0^6 + (-52i + 52)\,b_0^5 - 48b_0^4 + (-207i - 207)\,b_0^3 - 26ib_0^2 + (122i - 122)\,b_0 - 7)$.
Level 59. Dimension 36.		
III	4	$q + b_1q^2 + \left(\frac{1}{4}b_1^4 + \frac{9}{2}b_1^2 + \frac{65}{4}\right)q^3 + \left(b_1^2 + 4\right)q^4 + \left(\frac{1}{4}b_1^4 + \frac{11}{2}b_1^2 + \frac{93}{4}\right)q^5 + O(q^6)$, for $\Gamma_1(59)$ with nebentype mod 59 mapping $2 \mapsto -1$, with coefficients in $\mathbb{Q}[b_1]/(b_1^6 + 27b_1^4 + 215b_1^2 + 509)$.

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