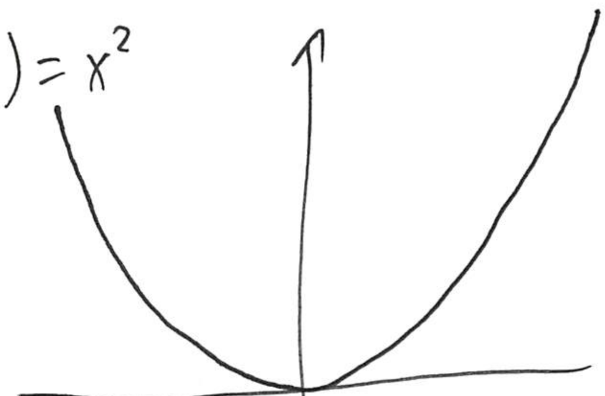
§2.5 Continuity

Idea: continuous functions have nice graphs

- no gaps, skips
- when you draw graph, pencil doesn't have to leave the paper

e.g. $f(x) = x^2$

1 ✓
2 ✓
3 ✓

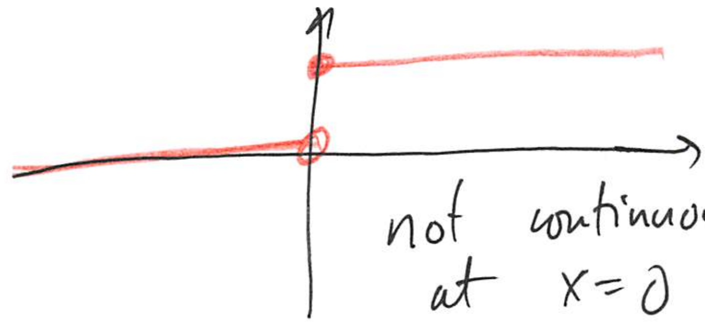


continuous for all its domain.

e.g. step function

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

1 ✓
2 X
3 ...



not continuous at $x = 0$

Def We say $f(x)$ is continuous at $x = a$ if

1. $f(a)$ is defined
2. $\lim_{x \to a} f(x) = L$ exists and is a number
3. $f(a) = L$

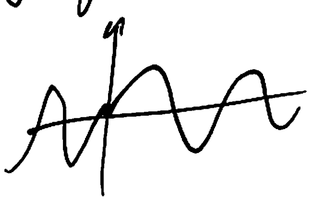
Many nice functions are continuous throughout their domains.

- polynomials
- rational functions $\frac{\text{poly.}}{\text{poly.}}$

  e.g. $\frac{x^2+1}{x-3}$
- algebraic functions

  built from polynomials, taking roots, ..

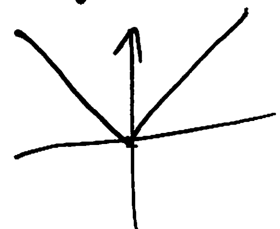
  e.g. $\sqrt{x^2+1}$
- trig fns $\sin x$, $\cos x$, $\tan x$, ...



- exponential functions

  e^x, 2^x, ...
- logarithmic functions

  $\ln x$, $\log_{10} x$, ...
- $|x|$



- sums $f+g$

  diffs $f-g$

  prods $f\cdot g$

  quotients f/g

  $\Bigg\}$ if f, g continuous $\Longrightarrow$ these also continuous.

e.g. $\frac{(x^2+|x|)\sin x}{1+\cos x}$

– composition of functions

$f(x)$, $g(x)$

$$(f \circ g)(x) = f(g(x))$$

e.g. $f(x) = e^x$

$g(x) = \sin x$

$$(f \circ g)(x) = e^{\sin x}$$

If f, g are continuous, so is $f \circ g$

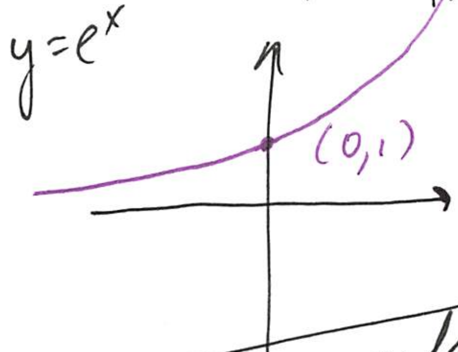
e.g. find C such that

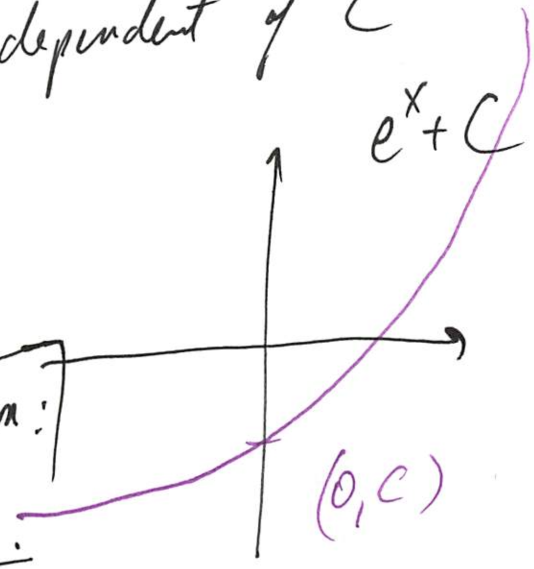
$$f(x) = \begin{cases} x^3 & x < 0 \\ e^x + C & x \geq 0 \end{cases}$$

is continuous, or explain why no such C exists.

If $x < 0$, f is continuous.

If $x > 0$, f is also continuous independent of C





$\boxed{\text{only poss. problem: } x = 0}$

at $x=0$:

① $f(0) = e^0 + C$
$= 1 + C$

② $\lim_{x\to 0} f(x) = ?$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x^3 = 0$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^x + C = 1+C$

need =

$\Rightarrow C = -1$

③ ✓

ANS: $C=-1$
makes f
continuous
everywhere

e.g. compute

$$\lim_{x\to\pi} \frac{\sin x}{1-\cos x}$$

Point: limit of a continuous function taken at a pt in the domain can be gotten by plugging in.

This fn. is continuous

$\sin x$, $\cos x$ continuous.

$1-\cos x$ cont.

$\frac{\sin x}{1-\cos x}$ ratio, cont.

$\cos\pi = -1$

$\Rightarrow 1-\cos\pi \neq 0$

$\Rightarrow \pi$ in domain of

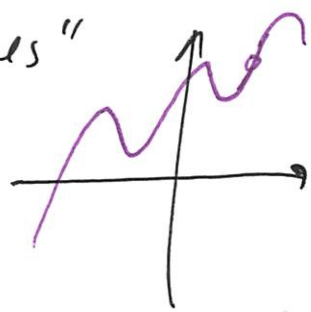
$$\frac{\sin x}{1-\cos x}$$

$$\Rightarrow \lim_{x\to\pi} \frac{\sin x}{1-\cos x} = \frac{\sin\pi}{1-\cos\pi}$$

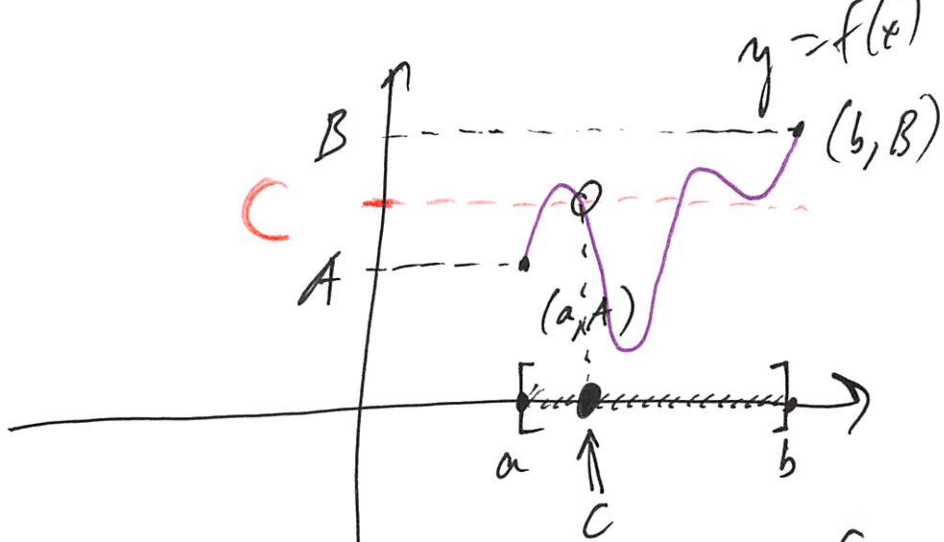
$$= \frac{0}{2} = \boxed{0}$$

<u>Intermediate value theorem</u>

"continuous functions don't skip values"



Suppose $f(x)$ is cont. for $a \le x \le b$. Suppose $f(a) = A$, $f(b) = B$. Then given any C between A and B, there exists $a \le c \le b$ such that $f(c) = C$



$f(c) = C$

application: finding roots of polynomials (or of other functions)

have $f(x)$, ~~wh~~ want a root: want α s.t. $f(\alpha) = 0$.

e.g. let $f(x) = x^3 - x^2 + 1$
show that $f(x)$ has at least one real root.

ANS: f is continuous.

$f(-10) = -1000 - 100 + 1 < 0$
$f(10) = 1000 - 100 + 1 > 0$

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IVThm $\Longrightarrow$ f has a root between $-10, 10$

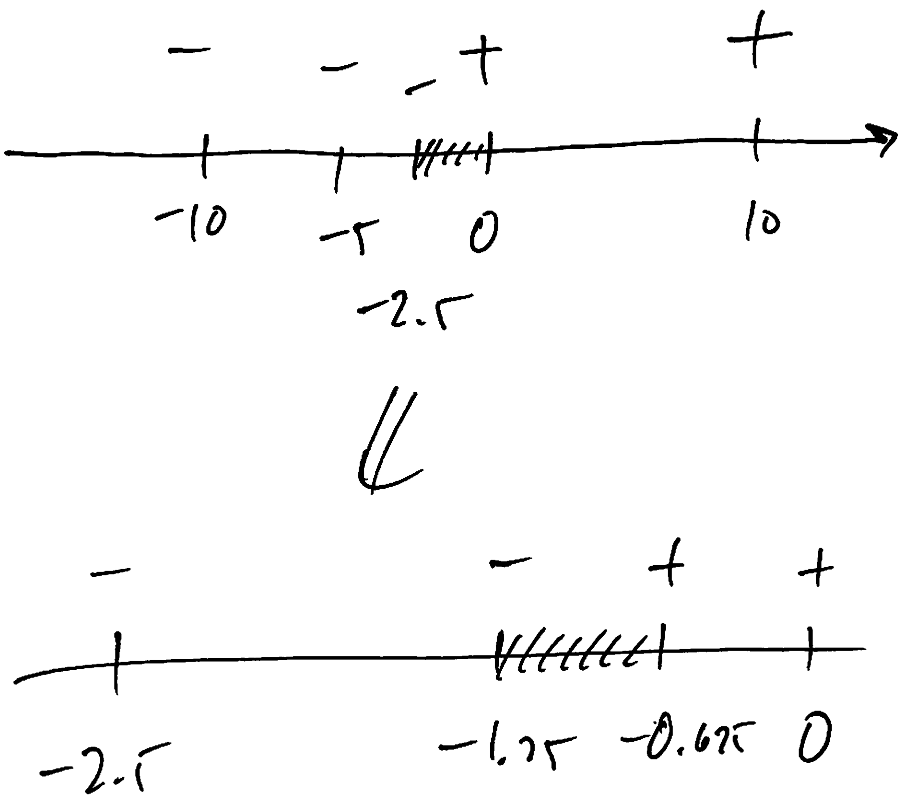
$a = -10$, $b = 10$

$C = 0$.

$\Longrightarrow$ Exists some c between $-10, 10$ s.t. $f(c) = 0$.

actual root is
$\alpha = -0.7548\ldots$

§2.6 limits at infinity



$\lim_{x\to a} f(x)$

can be a number, doesn't necc. exist, or could be $\pm\infty$.

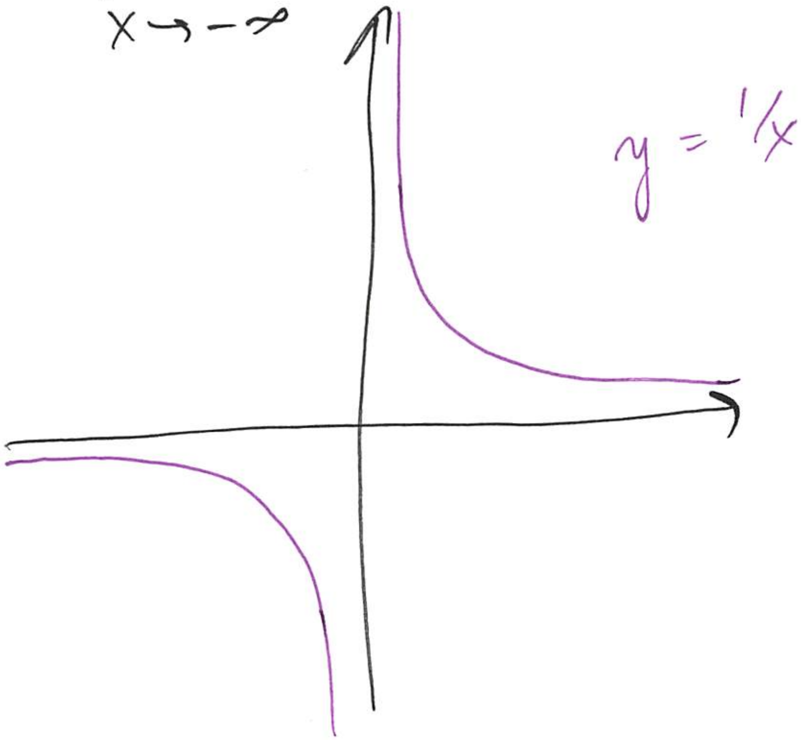
Now: $\lim_{x\to\infty} f(x)$

and $\lim_{x\to -\infty} f(x)$.

means: consider plugging in x's futher and further down the number line. If output of $f(x)$ stabilizes, we call that the value of the limit.

e.g $\lim_{x \to \infty} \frac{1}{x} = 0$

$\lim_{x \to -\infty} \frac{1}{x} = 0$



sim. $\lim_{x \to \infty} \frac{1}{x^2} = 0$ 8

$\lim_{x \to \infty} \frac{1}{x^3} = 0$

⋮

$\lim_{x \to \infty} \frac{1}{x^n} = 0$

$n > 0$

e.g $\frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$

e.g. $\lim\limits_{x\to\infty} \dfrac{2x^2+x+1}{x^2+3} = ?$

intuitively: x is big, get $\dfrac{2x^2}{x^2} \approx \dfrac{2}{1} = 2$

To do it correctly, we divide top & bottom by x^2.

$$\lim_{x\to\infty} \frac{(2x^2+x+1)\left(\frac{1}{x^2}\right)}{(x^2+3)\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{2 + \overset{\to 0}{\cancel{\frac{1}{x}}} + \overset{\to 0}{\cancel{\frac{1}{x^2}}}}{1 + \overset{\to 0}{\cancel{\frac{3}{x^2}}}}$$

$$= \boxed{2}$$

e.g. $\lim\limits_{x\to\infty} \dfrac{\left(\sqrt{x^4+1}\right)\frac{1}{x^2}}{\left(x^2+1\right)\frac{1}{x^2}}$

$$\cancel{= \lim_{x\to\infty} \frac{}{(x^2+1)}}$$

$\boxed{\sqrt{x^4} = x^2}$

$$\lim_{x\to\infty} \frac{\sqrt{1+\overset{\to 0}{\cancel{\frac{1}{x^4}}}}}{1+\overset{\to 0}{\cancel{\frac{1}{x^2}}}} = \frac{\sqrt{1}}{1} = \boxed{1}$$

e.g. $\lim\limits_{x \to -\infty} \dfrac{\sqrt{x^2+1}}{x+1}$

done similarly, but

ANS $= -1$, not 1.

use $\frac{1}{x}$, but

$\sqrt{x^2} \neq x$

in fact $\sqrt{x^2} = |x|$