MATH 131H EXAM I ANSWERS

This exam is worth 100 points, with each problem worth 20 points. **Please complete Problem (1) and any four of the remaining problems for a total of five problems.** There are problems on both sides of this page. Unless indicated otherwise, you must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

Please indicate on the cover of your exam booklet which problems you would like graded. Please indicate exactly five problems (including Problem 1); if you indicate more than five only the first five (in numerical order) including Problem 1 will be graded.

Let me know if you find any mistakes in the answers.

(1) (20 points) Please classify the following statements as **True** or **False**. Write out the word completely; do not simply write **T** or **F**. There is no partial credit for this problem, and it is not necessary to show your work for this problem.

(a) (4 points)

$$\lim_{x \to \pi} \cos(2x) = 0.$$

Answer: FALSE. $\cos(2\pi) = 1$.

(b) (4 points) The graph of

$$y = \frac{x+1}{x-1}$$

has a horizontal asymptote along the line $x = 1$. **Answer:** FALSE. The equation $x = 1$ is for a vertical asymptote, not a horizontal asymptote.

(c) (4 points) If $f(x) = (x+1)^3$, then $f'(x) = 6x^2 + 6x + 1$. **Answer:** FALSE. The derivative is $3x^2 + 6x + 3$.

(d) (4 points) Suppose $f(x)$ is a continuous function with domain all real numbers and that $f(-1) < 0$ and $f(1) > 0$. Then there is a c with $-1 < c < 1$ such that $f(c) = 0$. **Answer:** TRUE. This follows from the intermediate value theorem.

(e) (4 points) If $f'(a)$ exists, then $f(x)$ is continuous at $x = a$. **Answer:** TRUE. This is a fact about derivatives.

(2) (20 points) Please compute the following. It is not necessary to simplify derivatives in your answers.

Date: 11 October 2023.

(a) (5 points)

$$\lim_{x \to 2} \frac{x^2 - 5x + 6}{2x^2 - 3x - 2}$$

Answer: $-1/5$.

(b) (5 points)

$$\lim_{x \to -\infty} \frac{x^2 + 2x - 1}{3x^2 - x}$$

Answer: $1/3$.

(c) (5 points) $f'(x)$ if

$$f(x) = (x^5 + x + 3)(x^4 - x^2 + 1)$$

Answer: $(x^5 + x + 3)(4x^3 - 2x) + (5x^4 + 1)(x^4 - x^2 + 1)$.

(d) (5 points) $f'(x)$ if

$$f(x) = \frac{2x^2 + 1}{x + 1}$$

Answer:

$$\frac{(x + 1)(4x) - (2x^2 + 1)}{(x + 1)^2}.$$

(3) (20 points) Use the *definition* of the derivative (i.e. as a limit) to compute $f'(x)$ for the following functions. **Answer:** For these you use

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

To do the limit you have to use various techniques from class/HW, like simplifying fractions, multiplying top and bottom by the conjugate, etc.

(a) (10 points)

$$f(x) = \frac{1}{2x + 3}$$

Answer: $-2/(2x + 3)^2$

(b) (10 points)

$$f(x) = \sqrt{x - 4}$$

Answer:

$$\frac{1}{2}(x - 4)^{-1/2}$$

(4) (20 points) Let $f(x) = \dfrac{1}{x + 2}$. Then $f'(x) = -\dfrac{1}{(x + 2)^2}$.

(a) (10 points) Find an equation for the tangent line to the graph of $f(x)$ at the point $x = 0$. **Answer:** The point on the line is $(0, 1/2)$ and the slope is $f'(0) = -1/4$. An equation is therefore $y - 1/2 = -x/4$.

(b) (10 points) Find the all points on the graph of $f(x)$ where the tangent line is parallel to the line $x + y = 0$. **Answer:** Parallel lines have the same slope and are different lines. Call the original line L. The slope of L is -1, so we need to solve $f'(x) = -1$. This means $x = -1, -3$. The points are given by $(-1, 1)$ and $(-3, -1)$. (Note: *points* doesn't just mean x-coordinates.) We also have to check that the tangent lines are different from L to get it absolutely correct. (It could happen that a tangent line coincides with L, since that's a possibility when two linear equations have the same slope.) The line through $(-3, -1)$ is different from L (because $(-3, -1)$ doesn't satisfy $x + y = 0$) but $(-1, 1)$ does lie on L. So only the tangent line through $(-3, -1)$ gives a parallel line to L.

(5) (20 points) Is there a number a such that

$$\lim_{x \to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$$

exists? If so find a. **Answer:** The denominator factors as $(x - 1)(x + 2)$. Therefore the denominator is 0 when $x = -2$ and $x = 1$. So if we want the limit to exist the numerator will have to have one or more factors of $x + 2$ to cancel the factor of $x + 2$ from the denominator. This will happen if -2 is a root of the numerator. Plugging in $x = -2$ we get $12 - 2a + a + 3 = 0$ or $15 - a = 0$. So $a = 15$.

(6) (20 points) Let $f(x) = x|x|$.

(a) (6 points) What is the domain and range of $f(x)$? **Answer:** The domain is all x. If we use the multipart definition of $|x|$, we can get a different way to think about $f(x)$ that makes it easier to understand the range. We have

$$f(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

From this we can see that the range is all real numbers.

(b) (7 points) Sketch the graph of $f(x)$. **Answer:** From the multipart description in the first part, we see that the graph looks like a standard parabola when $x \geq 0$ and the parabola $y = -x^2$ when $x < 0$.

(c) (7 points) Show that $f'(0) = 0$ (Hint: use the definition of the derivative). **Answer:**

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{h|h| - 0}{h} \\ &= \lim_{h \to 0} |h| = 0 \end{aligned}$$

so $f'(0) = 0$.

(7) (20 points)

(a) (8 points) State the precise definition (in terms of ε, δ) of what it means for

$$\lim_{x \to a} f(x) = L.$$

Answer: A full statement of the definition goes *We say* $\lim_{x \to a} f(x) = L$ *if given any* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that if* $|x - a| < \delta$*, then* $|f(x) - L| < \varepsilon$*.* Note that it is more than just the formulas with ε and δ. The words matter too, like where the *if*s are, the *then*s are, which is given, which exists, etc.

(b) (12 points) Use the precise definition of a limit to verify that

$$\lim_{x \to 6} \left(\frac{x}{4} + 3 \right) = \frac{9}{2}.$$

Answer: We start with $|f(x) - L| < \varepsilon$ or $|\frac{x}{4} + 3 - \frac{9}{2}| < \varepsilon$. Then $\frac{1}{4}|x - 6| < \varepsilon$ or $|x - 6| \leq 4\varepsilon$. So if we are given $\varepsilon > 0$, as long as we take $\delta < 4\varepsilon$ we know that $|x - 6| < \delta$ will imply $|\frac{x}{4} + 3 - \frac{9}{2}| < \varepsilon$.