COHOMOLOGY OF LINE BUNDLES ON THE COTANGENT BUNDLE OF A GRASSMANNIAN

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To Professor Shoji, on the occasion of his 60th birthday.

ABSTRACT. We show that certain line bundles on the cotangent bundle of a Grassmannian arising from an anti-dominant character λ have cohomology groups isomorphic to those of a line bundle on the cotangent bundle of the dual Grassmannian arising from the dominant character $w_0(\lambda)$, where w_0 is the longest element of the Weyl group of $SL_{l+1}(k)$.

1. INTRODUCTION

Let k be an algebraically closed field of characteristic $p \geq 0$. Consider the algebraic group $G = SL_{l+1}(k)$. Let $T \subset B$ be a maximal torus contained in a Borel subgroup of G and let $X^*(T)$ denote the characters of T. We choose positive roots and simple roots Π in $X^*(T)$ which correspond to the Borel subgroup opposite to B. We index $\Pi = \{\alpha_j\}$ so that α_1 is an extremal root and α_j is next to α_{j+1} in the Dynkin diagram of type A_l. Let $\{\omega_i\}$ be the fundamental weights of G corresponding to Π. Let $\alpha^\vee$ be the coroot of the root α and let $\langle -, - \rangle$ denote the pairing of $X^*(T)$ and the cocharacters $X_*(T)$ of T.

For a rational representation V of B, denote by $H^*(G/B, V)$, or just $H^*(V)$ when there is no ambiguity, the cohomology of the sheaf of sections of the vector bundle $G \times^B V$ over G/B. For $\lambda \in X^*(T)$, we use the notation λ both for a character of T and for the one-dimensional representation of T or B that it defines.

Let P denote the maximal proper parabolic subgroup containing B corresponding to all the simple roots except α_m. Thus G/P identifies with the Grassmannian of m-planes in $l+1$-space. Let $\mathfrak{u}_m$ be the Lie agebra of the unipotent radical of P. Denote by $S^n\mathfrak{u}_m^*$ the n-th symmetric power of the linear dual of $\mathfrak{u}_m$.

The result of this paper is the following:

Theorem 1.1. *Let r be an integer in the range*

$$-|l+1-2m| - 1 \leq r \leq 0.$$

If $p = 0$ or

$$p > max\{r, min\{m, l+1-m\}\},$$

then there is a G-module isomorphism

$$H^i(G/B, S^n\mathfrak{u}_m^* \otimes r\omega_m) \simeq H^i(G/B, S^{n+rm}\mathfrak{u}_{l+1-m}^* \otimes -r\omega_{l+1-m}) \ \textit{for all} \ \ i, n \geq 0.$$

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That the theorem is related to the cohomology of line bundles on the cotangent bundle of a Grassmannian goes as follows. First, since all roots have the same length, the cotangent bundle of G/P identifies with the vector bundle $G \times^P \mathfrak{u}_m$. Second, for $\lambda \in X^*(P)$, let L_λ denote the corresponding line bundle on G/P. Let $\pi : G \times^P \mathfrak{u}_m \to G/P$ be the natural map. The line bundle L_λ can be pulled back to a line bundle $\pi^*(L_\lambda)$ on $G \times^P \mathfrak{u}_m$. We have

$$H^i(G \times^P \mathfrak{u}_m, \pi^*(L_\lambda)) \simeq \bigoplus_{n \geq 0} H^i(G/P, S^n \mathfrak{u}_m^* \otimes \lambda).$$

Finally,

$$H^i(G/B, V) \simeq H^i(G/P, V) \text{ for all } i \geq 0$$

for any P-representation V. See Chapter 8 of Jantzen's notes [3] or [1] for a discussion of these facts, which hold for any parabolic subgroup of a semisimple group.

Theorem 1.1 was used in the papers [4] and [5] to prove that certain nilpotent varieties are normal[1]. An analogous theorem for G of type D_{2l+1} was proved and used in [5], although that theorem was stated only in characteristic zero. The usefulness of Theorem 1.1 lies in the fact that it can be used for arbitrary semisimple G whenever P is replaced by a parabolic subgroup of G whose Levi subgroup L contains a semisimple subgroup M of type $A_{m-1} \times A_{l-m}$ such that G contains a Levi subgroup L' of semisimple type A_l where $M \subset L'$ and $[L', L] \subset L$. Then multiple applications of Theorem 1.1 often lead to a situation, at least in characteristic zero, where the cohomology groups vanish for all $n \geq 0$ when $i > 0$. The reason is that when $p = 0$ we are in a position to invoke a version of the Grauert-Riemenschneider theorem.

Perhaps the most interesting feature of the theorem is that it translates the cohomology of a line bundle on the cotangent bundle of one partial flag variety into the cohomology of a line bundle on the cotangent bundle of a *different* partial flag variety.

2. PRELIMINARIES

Recall the following proposition, due to Demazure and extended to positive characteristic by Thomsen. It is true for all semisimple groups, although we use it here only for type A. From now on, P_α refers to the minimal parabolic subgroup of G containing B corresponding to the simple root α. If $\alpha = \alpha_t$, then we may write P_t instead of P_{α_t}.

Proposition 2.1. [2] [6] *Let V be a rational representation of B and assume that V extends to a representation of the parabolic subgroup P_α. Let $\lambda \in X^*(T)$ be such that $s = \langle \lambda, \alpha^\vee \rangle \leq -1$. If $p = 0$ or $p > -s$, then there is a G-module isomorphism*

$$H^i(G/B, V \otimes \lambda) \simeq H^{i-1}(G/B, V \otimes \lambda - (s+1)\alpha) \text{ for all } i \geq 0.$$

In particular, if $s = -1$ then all cohomology groups $H^i(G/B, V \otimes \lambda)$ vanish.

This leads to the following result for $G = SL_{l+1}(k)$.

Lemma 2.2. *Let Q be a representation of B that extends to a representation of each P_t for $a \leq t \leq b$. Let $\lambda \in X^*(T)$ be such that $\langle \lambda, \alpha_t^\vee \rangle = 0$ for $a < t \leq b$. Set $s = \langle \lambda, \alpha_a^\vee \rangle$ and assume that $a - b - 1 \leq s \leq -1$. If $p = 0$ or $p > -s$, then $H^*(Q \otimes \lambda) = 0$.*

Proof. This is an application of Proposition 2.1, utilizing it a total of $-s$ times, starting with the parabolic P_a. After one application we have

$$H^i(Q \otimes \lambda) \simeq H^{i-1}(Q \otimes \lambda + (-s-1)\alpha_a).$$

After the second application we have that the latter is isomorphic to

$$H^{i-2}(Q \otimes \lambda + (-s-1)\alpha_a + (-s-2)\alpha_{a+1}).$$

[1]The reference for the theorem in those papers is supplanted by the current paper.

Continuing along, we find that after $-s-1$ times that

$$H^i(Q \otimes \lambda) \simeq H^{i+s+1}(Q \otimes \lambda + (-s-1)\alpha_a + \cdots + 2\alpha_{a-s-3} + \alpha_{a-s-2}).$$

This is possible since $a-s-2 < b$ and thus Q extends to a representation of P_j for $a \leq j \leq a-s-2 < b$.

We are done at this point because

$$\langle \lambda + (-s-1)\alpha_a + \cdots + \alpha_{a-s-2},\ \alpha^\vee_{a-s-1} \rangle = -1$$

and $a-s-1 \leq b$ so that Q is a representation of P_{a-s-1}. Thus Proposition 2.1 applies again, giving the total vanishing of cohomology. □

We can now prove the main result.

3. Proof of theorem 1.1

Proof. By symmetry we may assume that $m \leq l+1-m$, so that the extremal value for r is

$$-|l+1-2m| - 1 = 2m - l - 2.$$

Step 1. In this step, r may be an arbitrary integer. Consider the intersection $V = \mathfrak{u}_m \cap \mathfrak{u}_{\,l+1-m}$. We will show in Step 1 that for all i, n

$$H^i(S^n \mathfrak{u}^*_m \otimes r\omega_m) \simeq H^i(S^n V^* \otimes r\omega_m). \tag{1}$$

We begin by taking the Koszul resolution of the short exact sequence

$$0 \to U \to \mathfrak{u}^*_m \to V^* \to 0$$

(this defines U) and tensoring it with $r\omega_m$. This gives

$$0 \to \cdots \to S^{n-j}\mathfrak{u}^*_m \otimes \wedge^j U \otimes r\omega_m \to \cdots \to S^n\mathfrak{u}^*_m \otimes r\omega_m \to S^n V^* \otimes r\omega_m \to 0.$$

We claim that

$$H^*(S^{n-j}\mathfrak{u}^*_m \otimes \wedge^j U \otimes r\omega_m) = 0$$

for $1 \leq j \leq \dim U$ from which Equation 1 will follow. The T-weights of U are those of the form

$$\alpha_c + \alpha_{c+1} + \cdots + \alpha_d,$$

where $c \leq m$ and $m \leq d < l+1-m$. Therefore, if λ is a T-weight of $\wedge^j U$, there exists a with $m < a \leq l+1-m$ such that $-m \leq \langle \lambda, \alpha^\vee_a \rangle \leq -1$ and $\langle \lambda, \alpha^\vee_t \rangle = 0$ for $t > a$.

Set $s = \langle \lambda, \alpha^\vee_a \rangle$ and $b = l$. We can invoke Lemma 2.2 for λ and $Q := S^{n-j}\mathfrak{u}^*_m \otimes r\omega_m$. Indeed, Q is stable under the parabolic subgroups P_t for $t \geq a$. Also $a-b-1 \leq -m$ since $a \leq l+1-m$ and so s is in the range $a-b-1 \leq -m \leq s \leq -1$ and so the lemma applies, given our assumption on the characteristic of k. It follows that $H^*(Q \otimes \lambda) = 0$ for all weights λ appearing in $\wedge^j U$ for $1 \leq j \leq \dim U$. Thus if we filter $\wedge^j U$ by B-subrepresentations such that the consecutive quotients are one-dimensional, we deduce that $H^*(Q \otimes \wedge^j U) = 0$ for $1 \leq j \leq \dim U$ and Equation 1 follows.

Step 2.

Let $V_1 = V \cap \mathfrak{u}_{m-1}$ and $V_2 = V_1 \cap \mathfrak{u}_{\,l+2-m}$. If $m = 1$, then $\mathfrak{u}_{m-1}$ and $\mathfrak{u}_{l+2-m}$ are considered to be the zero vector space. Let μ be a weight of the form

$$r\omega_m + r'\omega_{l+1-m}$$

and assume that $2m-2-l \leq r \leq -1$ with r' unrestricted, unless $r = 2m-2-l$, in which case assume that $r' = 0$. In this step we show for all $n \geq 0$ that

$$H^*(S^n V^*_1 \otimes \mu) = 0 \tag{2}$$

Take the Koszul resolution of

$$0 \to U_2 \to V^*_1 \to V^*_2 \to 0$$

(this defines U_2) and tensor it with μ. We will show that

$$H^*(S^n V_2^* \otimes \mu) = 0$$

and

$$H^*(S^{n-j} V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0$$

for $1 \leq j \leq m-1$ and then Equation 2 will follow (the dimension of U_2 is $m-1$ as shown below).

The subspace V_2 coincides with $\mathfrak{u}_{m-1} \cap \mathfrak{u}_{\,l+2-m}$. Hence V_2^* is stable under P_t for $m \leq t \leq l+1-m$. It follows that $H^*(S^n V_2^* \otimes \mu) = 0$ by Lemma 2.2 with $a = m$, $b = l-m$ unless $r' = 0$ in which case $b = l+1-m$. In all cases, we have $a-b-1 \leq r \leq -1$ by hypothesis and the lemma applies since we are assuming $p > -r$.

Now the weights of U_2 are

$$\alpha_c + \alpha_{c+1} + \cdots + \alpha_{l+1-m}$$

where $1 \leq c \leq m-1$. If λ is a weight of $\wedge^j U_2$, then λ satisfies $\langle \lambda, \alpha_{l+2-m}^\vee \rangle = -j$ and $\langle \lambda, \alpha_t^\vee \rangle = 0$ for $t > l+2-m$. Consequently, if we filter $\wedge^j U_2$ as in Step 1 and apply Lemma 2.2, we get

$$H^*(S^{n-j} V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0$$

for $1 \leq j \leq m-1$. The lemma works with $a = l+2-m$, $b = l$. Thus $a-b-1 = (l+2-m)-l-1 = -m+1$ and j is in the acceptable range $-m+1 \leq -j \leq -1$. We are also using the fact that $S^{n-j} V_1^* \otimes \mu$ is stable under P_t for $t \geq l+2-m$.

Step 3. In this step, we show that for all i, n

$$H^i(S^n V^* \otimes \mu) \simeq H^i(S^{n-m} V^* \otimes \mu + \omega_m + \omega_{l+1-m}) \tag{3}$$

for μ as in Step 2.

We take the Koszul resolution of the short exact sequence

$$0 \to U_1 \to V^* \to V_1^* \to 0$$

(this defines U_1) and tensor it with μ to get

$$0 \to S^{n-m} V^* \otimes \wedge^m U_1 \otimes \mu \to \cdots \to S^{n-j} V^* \otimes \wedge^j U_1 \otimes \mu \to \cdots \to S^n V^* \otimes \mu \to S^n V_1^* \otimes \mu \to 0 \tag{4}$$

The weights of U_1 are of the form

$$\alpha_m + \alpha_{m+1} + \cdots + \alpha_d$$

where $l+1-m \leq d \leq l$ (and in particular, $\dim U_1 = m$). We study the terms $\wedge^j U_1$ for $1 \leq j < m$. If λ is a weight of $\wedge^j U_1$, then λ satisfies $\langle \lambda, \alpha_{m-1}^\vee \rangle = -j$ and $\langle \lambda, \alpha_t^\vee \rangle = 0$ for $t < m-1$. Consequently, proceeding as in Step 1, we filter $\wedge^j U_1$ and apply Lemma 2.2 (after applying an outer automorphism to G to arrive at the obvious symmetric set-up) to get

$$H^*(S^{n-j} V^* \otimes \wedge^j U_1 \otimes \mu) = 0$$

when $j < m$. The lemma works with $a = 1$, $b = m-1$, so that $a-b-1 = -m+1 \leq -j \leq -1$. We note that V is a representation of P_t for $t \leq m-1$.

On the other hand, for the case $j = m$, we have

$$\wedge^m U_1 = m(\alpha_m + \alpha_{m+1} + \cdots + \alpha_{l+1-m}) + (m-1)\alpha_{l+2-m} + \cdots + 2\alpha_{l-1} + \alpha_l.$$

So $m-1$ applications of Proposition Demazure as in the proof of Lemma 2.2 yields

$$H^i(S^{n-m} V^* \otimes \wedge^m U_1 \otimes \mu) \simeq H^{i+m-1}(S^{n-m} V^* \otimes \mu + \omega_m + \omega_{l+1-m}).$$

By breaking Equation 4 into short exact sequences and taking their cohomology, we conclude that

$$H^i(S^n V^* \otimes \mu) \simeq H^i(S^{n-m} V^* \otimes \mu + \omega_m + \omega_{l+1-m}),$$

where we are using

$$H^*(S^n V_1^* \otimes \mu) = 0$$

from Step 2.

Step 4. We obtain the theorem by using Step 3 repeatedly, starting with $\mu = r\omega_m$ with r in the prescribed range of the statement of the theorem. After $-r$ steps we arrive at

$$H^i(S^n V^* \otimes r\omega_m) \simeq H^i(S^{n+rm} V^* \otimes -r\omega_{l+1-m}),$$

for all i, n. The proof is completed by using Step 1 and the symmetric version of Equation 1 which gives

$$H^i(S^{n+rm} V^* \otimes -r\omega_{l+1-m}) \simeq H^i(S^{n+rm} \mathfrak{u}^*_{l+1-m} \otimes -r\omega_{l+1-m})$$

for all i, n.

□

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