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THE HOMOLOGY OF THE SPACE OF AFFINE FLAGS CONTAINING A NILPOTENT ELEMENT

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ABSTRACT. We show that the homology of the space of Iwahori subalgebras containing a nilpotent element of a split semisimple Lie algebra over $\mathbf{C}((\varepsilon))$ is isomorphic to the homology of the entire affine flag manifold.

1. INTRODUCTION

Let G be a semisimple, simply connected algebraic group over $\mathbf{C}$ with Lie algebra $\mathfrak{g}$. Let $F = \mathbf{C}((\varepsilon))$, $\hat{G} = G(F)$, and $\hat{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbf{C}} F$. The affine flag manifold of $\hat{G}$, denoted $\hat{\mathcal{B}}$, is the space of Iwahori subalgebras of $\hat{\mathfrak{g}}$. This is an infinite dimensional algebraic variety (i.e. a direct limit of complex varieties under closed embeddings) with a natural action of $\hat{G}$ given by conjugation.

We recall some facts about $\hat{\mathcal{B}}$. Let $\hat{\mathfrak{b}}_0$ be an Iwahori subalgebra and let $\hat{B}_0$ denote the subgroup of $\hat{G}$ which normalizes $\hat{\mathfrak{b}}_0$. Let $\hat{W}$ be the affine Weyl group of G. This is a Coxeter group with length function $l(w)$ for $w \in \hat{W}$ and standard partial order $\leq$. Viewing $\hat{W}$ as the quotient of a subgroup $\mathfrak{M}$ of $\hat{G}$ defined in [2], we denote by $\dot{w} \in \mathfrak{M}$ a representative of $w \in \hat{W}$. For each $w \in \hat{W}$, we let $\hat{\mathcal{B}}_w = \hat{B}_0 \dot{w} \hat{\mathfrak{b}}_0 \subset \hat{\mathcal{B}}$. Then $\hat{\mathcal{B}} = \bigcup \hat{\mathcal{B}}_w$, where the disjoint union is over all w in $\hat{W}$. This is the 'Bruhat' decomposition of $\hat{\mathcal{B}}$ [2]. Furthermore, $\hat{\mathcal{B}}_w$ is isomorphic to $\mathbf{C}^{l(w)}$ and $\overline{\hat{\mathcal{B}}_w}$ the closure of $\hat{\mathcal{B}}_w$ in $\hat{\mathcal{B}}$ is a complex projective variety which is the union of all $\hat{\mathcal{B}}_{w'}$ such that $w' \leq w$. Also $\hat{\mathcal{B}} = \varinjlim \overline{\hat{\mathcal{B}}_w}$ (see [4]).

Let N be a nilpotent element of $\hat{\mathfrak{g}}$. Let $\hat{\mathcal{B}}_N$ be the subspace of $\hat{\mathcal{B}}$ consisting of Iwahori subalgebras containing N. Our main result is that the singular homology with integer coefficients of $\hat{\mathcal{B}}_N$ is isomorphic to the homology of $\hat{\mathcal{B}}$.

2. KEY LEMMAS

Lemma 1. *Let $N \in \hat{\mathfrak{g}}$ be nilpotent. For any positive integer k, the elements N and $\varepsilon^{2k}N$ are conjugate under $\hat{G}$.*

Proof. Since $\hat{G}$ is split over F, the Jacobson-Morozov theorem [3] implies that N belongs to an $sl_2(F)$-subalgebra of $\hat{\mathfrak{g}}$. Now the lemma just follows from the case of $\hat{G} = SL_2(F)$.

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Thus for any positive integer k, we see that $\hat{\mathcal{B}}_N$ and $\hat{\mathcal{B}}_{\varepsilon^{2k}N}$ are homeomorphic.

Now we wish to put coordinates on the space $\hat{\mathcal{B}}_w = \hat{B}_0\dot{w}\hat{\mathfrak{b}}_0$ for $w \in \hat{W}$. First we need some more preliminaries. Choose $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Let Δ be the set of roots defined by $\mathfrak{h}$ and let $\Delta = \Delta^+ \cup \Delta^-$ be a decomposition of Δ into positive and negative roots. Let $\tilde{\Delta} = \{(\alpha, n) \,|\, \alpha \in \Delta \text{ and } n \in \mathbb{Z}\}$ and let $\tilde{\Delta}^+ = \{(\alpha, n) \in \tilde{\Delta} \,|\, \alpha \in \Delta^+ \text{ and } n > 0 \text{ or } \alpha \in \Delta^- \text{ and } n \geq 0\}$.

For $\alpha \in \Delta$, let e_α be a non-zero element in the root space of $\mathfrak{g}$ corresponding to α. For $t \in F$, let $x_\alpha(t) = \exp(te_\alpha)$ which is a well-defined element of $\hat{G}$. Now we fix $\hat{\mathfrak{b}}_0$ to be the Iwahori subalgebra spanned as a $\mathbf{C}$-vector space by $\mathfrak{h} \otimes \mathbf{C}[[\varepsilon]]$ and the elements $\varepsilon^n e_\alpha$ for all $(\alpha, n) \in \tilde{\Delta}^+$. Thus for $(\alpha, n) \in \tilde{\Delta}^+$ and $t \in \mathbf{C}$, we know that $x_\alpha(t\varepsilon^n) \in \hat{B}_0$.

Now let $\tilde{\Delta}(w) = \{(\alpha, n) \in \tilde{\Delta}^+ \,|\, \dot{w}^{-1}x_\alpha(\varepsilon^n)\dot{w} \notin \hat{B}_0\}$. It is known that the cardinality of $\tilde{\Delta}(w)$ is just $l(w)$ [2]. To simplify notation, let l equal $l(w)$. For $1 \leq j \leq l$, let (α_j, n_j) be the elements of $\tilde{\Delta}(w)$ in some order. Then from results in [2], it follows that the map $\phi : \mathbf{C}^l \to \hat{\mathcal{B}}_w$ given by

$$\phi(t_1, t_2, \ldots, t_l) \to x_{\alpha_1}(t_1\varepsilon^{n_1})x_{\alpha_2}(t_2\varepsilon^{n_2})\ldots x_{\alpha_l}(t_l\varepsilon^{n_l})\,\dot{w}\hat{\mathfrak{b}}_0$$

is an isomorphism. We use these coordinates in the following lemma.

Lemma 2. *Let $N \in \hat{\mathfrak{b}}_0$ be any element and let k be a positive integer. If $\hat{\mathcal{B}}_{\varepsilon^k N} \cap \hat{\mathcal{B}}_w \neq \emptyset$, then for some complex variety V and some $d \geq min\{k, l\}$, we have $\hat{\mathcal{B}}_{\varepsilon^k N} \cap \hat{\mathcal{B}}_w \simeq \mathbf{C}^d \times V$. Furthermore, $\hat{\mathcal{B}}_{\varepsilon^k N} \cap \hat{\mathcal{B}}_w = \hat{\mathcal{B}}_w$ for k sufficiently large.*

Proof. For $1 \leq j \leq l$, let $t_j \in \mathbf{C}$ and let $y = x_{\alpha_1}(t_1\varepsilon^{n_1})x_{\alpha_2}(t_2\varepsilon^{n_2})\ldots x_{\alpha_l}(t_l\varepsilon^{n_l})$ with the above notation so that $y\dot{w}\hat{\mathfrak{b}}_0$ is a point of $\hat{\mathcal{B}}_w$. We want to find the conditions on the t_j which specify when $y\dot{w}\hat{\mathfrak{b}}_0 \in \hat{\mathcal{B}}_{\varepsilon^k N}$.

It can be seen that any element $M \in \hat{\mathfrak{b}}_0$ can be written uniquely in the form $M = M^+ + M^-$ where $M^+ \in \hat{\mathfrak{b}}_0$, $\dot{w}^{-1}M^+\dot{w} \in \hat{\mathfrak{b}}_0$, and M^- is a $\mathbf{C}$-linear combination of $\varepsilon^n e_\alpha$ where $(\alpha, n) \in \tilde{\Delta}(w)$. Thus $\dot{w}^{-1}M\dot{w} \in \hat{\mathfrak{b}}_0$ if and only if $M^- = 0$. Since $y \in \hat{B}_0$ and $N \in \hat{\mathfrak{b}}_0$, we have $\varepsilon^k y^{-1}Ny \in \hat{\mathfrak{b}}_0$. Hence we can write $\varepsilon^k y^{-1}Ny$ in the above form as

$$M^+ + \sum_{(\alpha,n)\in\tilde{\Delta}(w)} p_{\alpha,n}\,\varepsilon^n e_\alpha.$$

Observe that $p_{\alpha,n}$ is a complex polynomial in the variables t_j for which $n_j \leq n - k$ and $p_{\alpha,n} = 0$ if $n < k$. This follows from the fact that if $t \in F$ and $\beta \in \Delta$, then $x_\beta(t)$ acting by conjugation on $\hat{\mathfrak{g}}$ acts as a matrix (with respect to a basis coming from $\mathfrak{g}$) with entries that are complex polynomials in t.

Now $y\dot{w}\hat{\mathfrak{b}}_0 \in \hat{\mathcal{B}}_{\varepsilon^k N}$ if and only if $\varepsilon^k\dot{w}^{-1}y^{-1}Ny\dot{w} \in \hat{\mathfrak{b}}_0$. And this is true if and only if $p_{\alpha,n} = 0$ for all $(\alpha, n) \in \tilde{\Delta}(w)$. Hence

$$\hat{\mathcal{B}}_{\varepsilon^k N} \cap \hat{\mathcal{B}}_w = \{(t_1, \ldots, t_l) \,|\, p_{\alpha,n} = 0 \text{ for all } (\alpha, n) \in \tilde{\Delta}(w)\}.$$

But $p_{\alpha,n}$ is not a function of all t_j. In fact, let $t_{j_1}, \ldots, t_{j_h}$ be a list of those t_j which appear in at least one $p_{\alpha,n}$. Let

$$V = \{(t_{j_1}, \ldots, t_{j_h}) \,|\, p_{\alpha,n} = 0 \text{ for all } (\alpha, n) \in \tilde{\Delta}(w)\}.$$

Thus if $V \neq \emptyset$, then $\hat{\mathcal{B}}_{\varepsilon^k N} \cap \hat{\mathcal{B}}_w \simeq \mathbf{C}^{l-h} \times V$.

Let $(\beta, m) \in \tilde{\Delta}(w)$ be such that $m = max\{\, n \,|\, (\alpha, n) \in \tilde{\Delta}(w)\}$. Then t_j does not contribute to any $p_{\alpha,n}$ if $n_j > m - k$. We now consider the two cases $0 < k \leq m$ and $k > m$ separately. If $0 < k \leq m$, then $(\beta, m), \ldots, (\beta, m-k+1)$ belong to $\tilde{\Delta}(w)$ but do not contribute to any $p_{\alpha,n}$. Hence in this case, the number of t_j such that $n_j > m - k$ is at least k, i.e., $l - h \geq k$. On the other hand, if $k > m$, then all $p_{\alpha,n}$ are zero, i.e., $\hat{\mathcal{B}}_{\varepsilon^k N} = \hat{\mathcal{B}}_w$. In either case, $\hat{\mathcal{B}}_{\varepsilon^k N} \cap \hat{\mathcal{B}}_w$ has the desired form and $\hat{\mathcal{B}}_{\varepsilon^k N} \cap \hat{\mathcal{B}}_w = \hat{\mathcal{B}}_w$ whenever $k > m$. This completes the proof.

3. Main Result

We return to the case where $N \in \hat{\mathfrak{g}}$ is nilpotent. We can now compute the singular homology of $\hat{\mathcal{B}}_N$ with integer coefficients.

Theorem. *For any nilpotent element $N \in \hat{\mathfrak{g}}$, we have $H_*(\hat{\mathcal{B}}_N) \simeq H_*(\hat{\mathcal{B}})$.*

Proof. Without loss of generality, we can assume $N \in \hat{\mathfrak{b}}_0$. Fix a nonnegative integer i. We wish to show that $H_i(\hat{\mathcal{B}}_N) \simeq H_i(\hat{\mathcal{B}})$. Since $\hat{\mathcal{B}}_N$ and $\hat{\mathcal{B}}_{\varepsilon^{2k} N}$ are homeomorphic by Lemma 1, it will be enough to show that $H_i(\hat{\mathcal{B}}_{\varepsilon^{2k} N}) \simeq H_i(\hat{\mathcal{B}})$ for some positive integer k.

By Lemma 2, we can choose k so that for all $w \in \hat{W}$ with $l(w) \leq i$, we have $\hat{\mathcal{B}}_{\varepsilon^{2k} N} \cap \hat{\mathcal{B}}_w = \hat{\mathcal{B}}_w$. Furthermore, by increasing k if necessary, we can also guarantee that if $l(w) > i$, then either $\hat{\mathcal{B}}_{\varepsilon^{2k} N} \cap \hat{\mathcal{B}}_w = \emptyset$ or for some complex variety V, we have $\hat{\mathcal{B}}_{\varepsilon^{2k} N} \cap \hat{\mathcal{B}}_w \simeq \mathbf{C}^{i+1} \times V$.

Let us introduce some more notation. For $w \in \hat{W}$, let

$$Y_w = \hat{\mathcal{B}}_{\varepsilon^{2k} N} \cap \hat{\mathcal{B}}_w$$

and

$$X_n = \bigcup_{\substack{w \in \hat{W} \\ l(w) \leq n}} Y_w.$$

Note that X_n is closed in $\hat{\mathcal{B}}_{\varepsilon^{2k} N}$ and $\hat{\mathcal{B}}_{\varepsilon^{2k} N} = \varinjlim X_n$ as topological spaces. Clearly

$$X_n = X_{n-1} \cup Y_{w_1} \cup Y_{w_2} \cup \cdots \cup Y_{w_h}$$

where $\{w_1, w_2, \ldots, w_h\}$ are the elements of $\hat{W}$ of length n. Let $X_{n,0} = X_{n-1}$ and for $1 \leq j \leq h$, define $X_{n,j}$ inductively to be $X_{n,j-1} \cup Y_{w_j}$. Thus $X_{n,h} = X_n$. Note that $X_{n,j-1}$ is closed in $X_{n,j}$.

Because k is chosen so that $Y_w = \hat{\mathcal{B}}_w$ for all $w \in \hat{W}$ with $l(w) \leq i$, we see that for $0 \leq r \leq 2i$

$$H_r(X_i) \simeq H_r(\hat{\mathcal{B}}).$$

In particular, this holds for $r = i$.

Now suppose we can establish that for all $n \geq i+1$ the inclusion of X_{n-1} in X_n induces an isomorphism of $H_i(X_{n-1})$ with $H_i(X_n)$. Then since $\hat{\mathcal{B}}_{\varepsilon^{2k} N} = \varinjlim X_n$, it will follow that

$$H_i(\hat{\mathcal{B}}_{\varepsilon^{2k} N}) \simeq H_i(X_i) \simeq H_i(\hat{\mathcal{B}}),$$

which would complete the argument.

So assume $n \geq i+1$. Let $H_*^{BM}(X)$ denote the Borel-Moore homology with integer coefficients of the space X [1]. Then the following exact sequence holds

$$\cdots \longrightarrow H_{i+1}^{BM}(Y_{w_j}) \longrightarrow H_i^{BM}(X_{n,j-1}) \longrightarrow H_i^{BM}(X_{n,j}) \longrightarrow H_i^{BM}(Y_{w_j}) \longrightarrow \cdots$$

since $X_{n,j-1}$ is closed in $X_{n,j}$. By the assumption on n and the choice of k, we have either $Y_{w_j} = \emptyset$ or $Y_{w_j} \simeq \mathbf{C}^{i+1} \times V$. In the latter case, since $H_r^{BM}(\mathbf{C}^{i+1})$ vanishes for $r < 2(i+1)$, the Künneth theorem implies that both $H_{i+1}^{BM}(Y_{w_j})$ and $H_i^{BM}(Y_{w_j})$ vanish. Hence in either case, the inclusion of $X_{n,j-1}$ in $X_{n,j}$ induces an isomorphism of $H_i^{BM}(X_{n,j-1})$ with $H_i^{BM}(X_{n,j})$. Since this holds for all j, we have the desired isomorphism of $H_i^{BM}(X_{n-1})$ with $H_i^{BM}(X_n)$ for all $n \geq i+1$. Finally, we note that the isomorphism is valid in singular homology because the X_n are compact and triangulable (being projective), so there is no distinction between Borel-Moore and singular homology.

4. Generalization

A similar result holds for the partial affine flag manifolds. Let $\hat{\mathfrak{p}}_0$ be a parahoric subalgebra of $\hat{\mathfrak{g}}$ and let $\hat{\mathcal{P}}$ be the space of subalgebras conjugate to $\hat{\mathfrak{p}}_0$. For a nilpotent element $N \in \hat{\mathfrak{g}}$, we let $\hat{\mathcal{P}}_N$ be the subspace of $\hat{\mathcal{P}}$ consisiting of parahoric subalgebras conjugate to $\hat{\mathfrak{p}}_0$ which contain N. Then $H_*(\hat{\mathcal{P}}_N) \simeq H_*(\hat{\mathcal{P}})$. The proof is similar to the above case.

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