

COMPONENT GROUPS OF UNIPOTENT CENTRALIZERS IN GOOD CHARACTERISTIC

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To Robert Steinberg, on his 80th birthday.

ABSTRACT. Let G be a connected, reductive group over an algebraically closed field of good characteristic. For $u \in G$ unipotent, we describe the conjugacy classes in the component group $A(u)$ of the centralizer of u . Our results extend work of the second author done for simple, adjoint G over the complex numbers.

When G is simple and adjoint, the previous work of the second author makes our description combinatorial and explicit; moreover, it turns out that knowledge of the conjugacy classes suffices to determine the group structure of $A(u)$. Thus we obtain the result, previously known through case-checking, that the structure of the component group $A(u)$ is independent of good characteristic.

Throughout this note, G will denote a connected and reductive algebraic group G over the algebraically closed field k . For the most part, the characteristic $p \geq 0$ of k is assumed to be *good* for G (see §1 for the definition).

The main objective of our note is to extend the work of the second author [Som98] describing the component groups of unipotent (or nilpotent) centralizers. We recall a few definitions before stating the main result.

A *pseudo-Levi* subgroup L of G is the connected centralizer $C_G^o(s)$ of a semisimple element $s \in G$. The reductive group L contains a maximal torus T of G , and so L is generated by T together with the 1 dimensional unipotent subgroups corresponding to a subsystem R_L of the root system R of G ; in §9 we make explicit which subsystems R_L arise in this way when G is quasisimple.

Let $u \in G$ be a unipotent element, and let $A(u) = C_G(u)/C_G^o(u)$ be the group of components (“component group”) of the centralizer of u . We are concerned with the structure of the group $A(u)$ (more precisely: with its conjugacy classes).

Consider the set of all triples

$$(1) \quad (L, tZ^o, u)$$

where L is a pseudo-Levi subgroup with center $Z = Z(L)$, the coset $tZ^o \in Z/Z^o$ has the property that $L = C_G^o(tZ^o)$, and $u \in L$ is a distinguished unipotent element.

Theorem 1. *Let G be connected and reductive in good characteristic. The map*

$$(L, sZ^o, u) \mapsto (u, sC_G^o(u))$$

yields a bijection between: G -conjugacy classes of triples as in (1), and G -conjugacy classes of pairs (u, x) where $u \in G$ is unipotent and x is an element in $A(u)$.

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The theorem is proved, after some preliminaries, in §8.

Remark 2. The G -conjugacy classes of pairs (u, x) as in the statement of the theorem are in obvious bijection with G -conjugacy classes of pairs (u, C) where $u \in G$ is unipotent and $C \subset A(u)$ is a conjugacy class.

Remark 3. Assume that G is simple and adjoint. We show in §10 that our work indeed extends the results of the second author. If $u \in G$ is unipotent, we find as a consequence of Theorem 1 that the conjugacy classes in $A(u)$ are in bijection with $C_G(u)$ -conjugacy classes of pseudo-Levi subgroups L containing u as a distinguished unipotent element; this was proved for $k = \mathbf{C}$ in [Som98]. It follows that $A(u) \simeq A(\hat{u})$ where \hat{u} is a unipotent element in the corresponding group over \mathbf{C} with the same labeled diagram as u . This isomorphism was known previously by case-checking in the exceptional groups; see especially [Miz80]. The structure of $A(u)$ for the exceptional groups when $k = \mathbf{C}$ is originally due to Alekseevski [Ale79].

Remark 4. Our proof of Theorem 1 is free of case-checking, with the following caveats. We use Pommerening's proof of the Bala-Carter theorem (specifically, we use the construction of "associated co-characters" for unipotent elements) in the proof of Proposition 12. Moreover, we use work of Premet to find Levi factors in the centralizer of a unipotent element; see Proposition 10.

The authors would like to thank the referee for pointing out an oversight and suggesting the use of Jantzen's result (Proposition 22) to prove Proposition 23. Upon completion of this paper, we learned that Premet has also given a case-free proof of Theorem 36.

1. REDUCTIVE ALGEBRAIC GROUPS

Fix $T \subset B \subset G$, where T is a maximal torus and B a Borel subgroup. Let (X, R, Y, R^\vee) denote the root datum of the reductive group G with respect to T ; thus $X = X^*(T)$ is the character group, and $R \subset X$ is the set of weights of T on \mathfrak{g} . Fix $S \subset R$ a system of simple roots.

When R is irreducible, the root with maximal height (with respect to S) will be denoted $\tilde{\alpha}$. Write

$$(2) \quad \tilde{\alpha} = \sum_{\beta \in S} a_\beta \beta$$

for positive integers a_β . The characteristic p of k is said to be good for G (or for R) if p does not divide any a_β . So $p = 0$ is good, and we may simply list the bad (i.e. not good) primes: $p = 2$ is bad unless $R = A_r$, $p = 3$ is bad if $R = G_2, F_4, E_r$, and $p = 5$ is bad if $R = E_8$.

The prime p is good for a general R just in case it is good for each irreducible component of R .

For a root $\alpha \in R$, let $\mathbf{G}_\alpha \simeq \mathcal{X}_\alpha \subset G$ be the corresponding root subgroup.

2. SPRINGER'S ISOMORPHISM

Let $\mathcal{U} \subset G$ and $\mathcal{N} \subset \mathfrak{g}$ denote respectively the unipotent and nilpotent varieties. In characteristic 0, the exponential is a G -equivariant isomorphism $\mathcal{N} \rightarrow \mathcal{U}$; in good characteristic, one has the following substitute for the exponential:

Proposition 5. *There is a G -equivariant homeomorphism $\varepsilon : \mathcal{N} \rightarrow \mathcal{U}$. Moreover, if R has no component of type A_r for which $r \equiv -1 \pmod{p}$, there is such an ε which is an isomorphism of varieties.*

Proof. There is an isogeny $\pi : \tilde{G} \rightarrow G$ where $\tilde{G} = \prod_i G_i \times T$ with T a torus and each G_i a simply connected, quasisimple group; see e.g. [Spr98, Theorem 9.6.5]. Let $\tilde{\mathcal{N}}$ and $\tilde{\mathcal{U}}$ denote the corresponding varieties for \tilde{G} . Since the characteristic is good, it has been proved by Springer [Spr69] that there is a \tilde{G} -equivariant isomorphism of varieties $\tilde{\varepsilon} : \tilde{\mathcal{N}} \rightarrow \tilde{\mathcal{U}}$; for another proof, see [BR85].

It follows from [McN, Lemma 24] that π restricts to a homeomorphism $\pi|_{\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \rightarrow \mathcal{U}$, and that $d\pi$ restricts to a homeomorphism $d\pi|_{\tilde{\mathcal{N}}} : \tilde{\mathcal{N}} \rightarrow \mathcal{N}$. Since the characteristic is good, $d\pi$ is bijective provided that $R \neq A_r$ when $r \equiv -1 \pmod{p}$; see the summary in [Hum95, 0.13]. It follows from the remaining assertion in [McN, Lemma 24] that $\pi|_{\tilde{\mathcal{U}}}$ and $d\pi|_{\tilde{\mathcal{N}}}$ are isomorphisms of varieties when $d\pi$ is bijective, whence the proposition. \square

In what follows, we fix an equivariant homeomorphism $\varepsilon : \mathcal{N} \rightarrow \mathcal{U}$, to which we will refer without further comment.

3. ASSOCIATED CO-CHARACTERS

Recall that a unipotent $u \in G$ is said to be *distinguished* if the connected center $Z^\circ(G)$ of G is a maximal torus of $C_G(u)$. A nilpotent element $X \in \mathfrak{g}$ is then distinguished if $\varepsilon(X)$ has that property.

Let $X \in \mathfrak{g}$ be nilpotent. If X is not distinguished, there is a Levi subgroup L of G for which $X \in \text{Lie}(L)$ is distinguished.

A co-character $\phi : k^\times \rightarrow G$ is said to be associated to X if

$$\text{Ad } \phi(t)X = t^2 X \quad \text{for each } t \in k^\times,$$

and if the image of ϕ lies in the derived group of some Levi subgroup L for which $X \in \text{Lie}(L)$ is distinguished.

A co-character ϕ is associated to a unipotent $u \in G$ if it is associated to $X = \varepsilon^{-1}(u)$.

Proposition 6. *Let $u \in G$ be unipotent. Then there exist co-characters associated to u , and any two such are conjugate by an element of $C_G^\circ(u)$.*

Proof. This is proved in [Jan, Lemma 5.3]. \square

Remark 7. The existence of associated co-characters asserted in the the previous proposition relies in an essential way on Pommerening's proof [Pom] of the Bala-Carter theorem in good characteristic.

Let ϕ be a co-character associated to the unipotent $u \in G$, and let $\mathfrak{g}(i)$ be the i -weight space for $\text{Ad} \circ \phi(k^\times)$, $i \in \mathbf{Z}$. Let $\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}(i)$. Then $\mathfrak{p} = \text{Lie}(P)$ for a parabolic subgroup P of G ; P is known as the canonical parabolic associated with u .

Proposition 8. *Let $u \in G$ be unipotent. The parabolic subgroup P is independent of the choice of associated co-character ϕ for u . Moreover, $C_G(u) \leq P$.*

Proof. [Jan, Prop. 5.9] \square

Remark 9. The proof that $C_G(u) \subset P$ is somewhat subtle in positive characteristic. Let $X = \varepsilon^{-1}(u)$. In characteristic 0, the assertion $C_G^o(u) \subset P$ is a consequence of the Lie algebra analogue $\mathfrak{c}_{\mathfrak{g}}(X) \subset \mathfrak{p}$ which follows from the Jacobson-Morozov theorem. (The fact that the full centralizer lies in P is then a consequence of the unicity of the canonical parabolic P). In good characteristic, the required assertion for the Lie algebra was proved by Spaltenstein, and independently by Premet; see the references in [Jan, §5]. In the positive characteristic case, the transition to the group is more subtle; again see *loc. cit.*

4. THE LEVI DECOMPOSITION OF A UNIPOTENT CENTRALIZER

In characteristic $p > 0$, a linear algebraic group can fail to have a Levi decomposition. Moreover, even when they exist, two Levi factors need not in general be conjugate. If $u \in G$ is unipotent and the characteristic is good for G , the connected centralizer $C_G^o(u)$ does have a Levi decomposition, thanks to work of Premet. More precisely:

Proposition 10. *Let $u \in G$ be unipotent, let P be the canonical parabolic associated with u (see Proposition 8), and let U_P be the unipotent radical of P .*

- (1) $R(u) = C_G(u) \cap U_P$ is the unipotent radical of $C_G(u)$.
- (2) For any co-character ϕ associated with u , the centralizer C_ϕ of ϕ in $C_G(u)$ is a Levi factor of $C_G(u)$; i.e. C_ϕ is reductive and $C_G(u) = C_\phi \cdot R(u)$.
- (3) If ϕ, ϕ' are two co-characters associated to u , then C_ϕ and $C_{\phi'}$ are conjugate by an element in $C_G^o(u)$.

Proof. [Jan, §5.10, 5.11]. □

Remark 11. The proof that $R(u)$ is a connected (normal, unipotent) group is elementary, as is the fact that $C_G(u) = C_\phi \cdot R(u)$. The proof that C_ϕ is reductive depends on work of Premet, and ultimately involves case-checking in small characteristics for exceptional groups.

5. SEMISIMPLE REPRESENTATIVES

If H is a linear algebraic group, in characteristic 0 one may always represent a coset $tH^o \in H/H^o$ by a semisimple element $t \in H$. In characteristic $p > 0$ this is no longer true in general (e.g. if $[H : H^o] \equiv 0 \pmod{p}$).

Let now G be connected, reductive in good characteristic and suppose $u \in G$ is unipotent. Despite the above difficulty, we may always choose semisimple representatives for the elements in the component group $A(u)$.

Proposition 12. *Let $u \in G$ be unipotent, and suppose $v \in C_G(u)$ is also unipotent. Then $v \in C_G^o(u)$.*

Proof. The proposition follows from [SS70, III.3.15]. Note that in *loc. cit.* G is assumed semisimple, but the argument works for all reductive G in view of Proposition 5. □

Corollary 13. *Let $u \in G$ be unipotent. Then each element of the component group $A(u)$ may be represented by a semisimple element $s \in C_G(u)$.*

Proof. Let $g \in C_G(u)$, and let $g = g_s g_u$ be its Jordan decomposition where g_s is semisimple and g_u is unipotent. Proposition 12 implies that $g_u \in C_G^o(u)$, whence the corollary. □

6. PSEUDO-LEVI SUBGROUPS

We collect here a few results on pseudo-Levi subgroups which will be needed in the proof of Theorem 1. Recall that by a pseudo-Levi subgroup, we mean the connected centralizer of a semisimple element of G .

Lemma 14. *Let $S \subset T$ be a subset. Then $C_G^\circ(S)$ is a reductive subgroup of G , and is generated by T together with the root subgroups \mathcal{X}_α for which $\alpha(S) = 1$.*

Proof. [SS70, II §4.1]. □

Proposition 15. *Let $L = C_G^\circ(t)$ with $t \in G$ semisimple. Write Z for the center of L .*

- (1) $L = C_G^\circ(tZ^\circ)$.
- (2) *Let S be a torus in $C_G^\circ(t)$, and let $M = C_G^\circ(tS)$. There is a non-empty open subset $U \subseteq tS$ such that $M = C_G^\circ(x)$ for each $x \in U$. In particular, M is again a pseudo-Levi subgroup of G . If Z_1 denotes the center of M , then $M = C_G^\circ(tZ_1^\circ)$.*
- (3) *There is a non-empty open subset $U \subseteq tZ^\circ$ such that $L = C_G^\circ(x)$ for each $x \in U$.*

Proof. (1) is straightforward to verify.

For (2), we may suppose that t and S are in T . Let $R' = \{\alpha \in R \mid \alpha(tS) = 1\}$. Then $R' \subseteq R_x = \{\alpha \in R \mid \alpha(x) = 1\}$ for any $x \in tS$. Since tS is an irreducible variety, the intersection of non-empty open subsets

$$U = \bigcap_{\alpha \in R \setminus R'} \{x \in tS \mid \alpha(x) \neq 1\}$$

is itself open and non-empty; moreover, it is clear that $R_x = R'$ whenever $x \in U$, so the first assertion of (2) follows from Lemma 14.

For the final assertion of (2), first note that $M = C_G^\circ(t, S) = C_L^\circ(S)$ is a Levi subgroup of L . By [DM91, Prop. 1.21] we have $M = C_L^\circ(Z_1^\circ)$; since t is central in M , we have also $M = C_L^\circ(tZ_1^\circ)$. Since certainly $C_G^\circ(tZ_1^\circ) \subseteq C_G^\circ(t) = L$, we deduce that $M = C_G^\circ(tZ_1^\circ)$ as desired.

(3) follows from (1) and (2) with $S = Z^\circ$. □

Proposition 16. *Let G be connected and reductive. If the characteristic p of k is good for G , and if L is a pseudo-Levi subgroup of G , then p is good for L as well.*

Proof. As in the proof of Proposition 5, let $\pi : \tilde{G} \rightarrow G$ be an isogeny where $\tilde{G} = \prod_i G_i \times S$ with S a torus and each G_i a simply connected quasisimple group. Let $L = C_G^\circ(t)$. If $\pi(\tilde{t}) = t$ and $\tilde{L} = C_{\tilde{G}}^\circ(\tilde{t})$, then Lemma 14 shows that $\pi(\tilde{L}) = L$. Since p is good for L if and only if it is good for \tilde{L} , we may replace G by \tilde{G} . Since $L = \prod_i (L \cap G_i) \times S$, it suffices to suppose that G is quasisimple.

According to [SS70, §4.1.4.3] p is good for G if and only if $\mathbf{Z}R/\mathbf{Z}R_1$ has no p -torsion for any (integrally) closed subsystem R_1 of R . Since the root system R_L of L is one such subsystem, it readily follows that p is good for any irreducible component of R_L . □

Lemma 17. *Let L be a pseudo-Levi subgroup of G . Then $L = C_G^\circ(s)$ for a semisimple element $s \in G$ of finite order.*

Proof. Let Z denote the (full) center of L . By [Spr98, Exerc. 3.2.10 5(b)], the elements of Z which have finite order are dense in the diagonalizable group Z . Now choose $t \in Z$ such that $L = C_G^\circ(t)$, and let $U \subset tZ^\circ$ be an open set as in Proposition 15(3). Then U is also open in Z and hence contains an element s of finite order. □

7. SEMISIMPLE AUTOMORPHISMS OF REDUCTIVE GROUPS

If H is any linear algebraic group, an automorphism σ of H is semisimple if there is a linear algebraic group H' with $H \triangleleft H'$ such that $\sigma = \text{Int}(x)|_H$ for some semisimple $x \in H'$ (where $\text{Int}(x)$ denotes the inner automorphism determined by x).

Proposition 18. *Let H be a connected linear algebraic group, and let σ be a semisimple automorphism of H . Then σ fixes a Borel subgroup B of H , and a maximal torus $T \subset B$.*

Proof. [Ste68, Theorem 7.5]. \square

Lemma 19. *Let A be a connected commutative linear algebraic group, let σ be a semisimple automorphism of A , and let A^σ be the fixed points of σ on A . Then each element $a \in A$ can be written*

$$a = x \cdot \sigma(y)y^{-1}$$

for $x \in A^\sigma$ and $y \in A$.

Proof. The homomorphism $\phi : A^\sigma \times A \rightarrow A$ given by $\phi(x, y) = x \cdot \sigma(y)y^{-1}$ has surjective differential by [Spr98, Corollary 5.4.5(ii)], so ϕ is surjective. \square

Proposition 20. *Let H be a reductive algebraic group, and suppose the images of the semisimple elements $t, t' \in H$ lie in the same conjugacy class in H/H° . Then there is $g \in H$ and a semisimple $s \in C_H^\circ(t)$ such that $gt'g^{-1} = ts$.*

Proof. Replacing t' by $ht'h^{-1}$ for suitable $h \in H$, we can suppose that t and t' have the same image in H/H° .

Applying Proposition 18 we can find $T \subset B$ where T and B are respectively an $\text{Int}(t)$ -stable maximal torus and Borel group. Similarly, we can find an $\text{Int}(t')$ -stable $T' \subset B'$.

Choose $g \in H$ with $B = g^{-1}B'g$. Then $g^{-1}T'g$ is a sub-torus of B . Replacing g by bg for some $b \in B$, we can arrange that $g^{-1}T'g = T$; replacing t' by $gt'g^{-1}$, we see that $T \subset B$ is both $\text{Int}(t)$ -stable and $\text{Int}(t')$ -stable.

Thus $n = t^{-1}t'$ is in the normalizer of T in H° . Since $\text{Int}(n)$ fixes B , and since the Weyl group $N_{H^\circ}(T)/T$ acts simply transitively on the set of Borel subgroups containing T , we deduce that $n \in T$. We can therefore write $t' = ta$ for $a \in T$. By Lemma 19 we can write $a = xt^{-1}yty^{-1}$ for some $x \in C_T(t)$ and $y \in T$. Let $g = ty^{-1}$. Then one readily checks that

$$gt'g^{-1} = tx$$

and the proof is complete. \square

Corollary 21. *Let H be a linear algebraic group. Suppose that \mathcal{M} is a collection of Levi factors of H which are all conjugate under H . If the semisimple elements $t, t' \in H$ lie in the same conjugacy class in H/H° , and if $t, t' \in \cup_{M \in \mathcal{M}} M$, then there is $g \in H$ and a semisimple element $s \in C_H^\circ(t)$ such that $gt'g^{-1} = ts$.*

Proof. Choose $M, M' \in \mathcal{M}$ with $t \in M$ and $t' \in M'$. Since M and M' are H -conjugate, replacing t' by an H -conjugate permits us to suppose that $t, t' \in M$. Since M is reductive, we deduce the result from Proposition 20. \square

We require one further property of pseudo-Levi subgroups, whose proof depends on Proposition 18 and on the following version of a result of Mostow recently obtained by Jantzen [Jan, 11.24]:

Proposition 22. *Let Γ be an algebraic group which is a semidirect product of a (not necessarily connected) reductive group M and a normal unipotent group R . Let $H \leq \Gamma$ be a linearly reductive closed subgroup of Γ . Then there exists $r \in R$ with $rHr^{-1} \subset M$.*

Proposition 23. *Let L be a pseudo-Levi subgroup of G and $u \in L$ a distinguished unipotent element. If a cocharacter of L is associated to u in L , then that cocharacter is associated to u in G as well.*

Proof. Since all cocharacters associated to u in L are conjugate by $C_L^o(u)$ (Proposition 6), it suffices to find some cocharacter of L which is associated to u in both L and G .

According to Lemma 17, $L = C_G^o(s)$ for some semisimple element s of finite order. The order of s is then invertible in k , so the cyclic subgroup H generated by s is linearly reductive (all of its linear k -representations are completely reducible).

Let ϕ be any cocharacter of G associated to u , and consider the subgroup $N = \phi(k^\times)C_G(u)$ (i.e. the group generated by the centralizer, and by the image of ϕ ; this is the group $N(\varepsilon^{-1}(u))$ defined in [Jan, 2.10(2)]).

According to Proposition 10, the centralizer C_ϕ of ϕ in $C_G(u)$ is a Levi factor of $C_G(u)$. Now $C'_\phi = \phi(k^\times) \cdot C_\phi$ is a Levi factor of N . Moreover, the image of ϕ is central in C'_ϕ .

Now take $\Gamma = N$ in Proposition 22. Then $H = \langle s \rangle$ is a linearly reductive subgroup of Γ . So there is an element r in the unipotent radical of $C_G(u)$ such that rsr^{-1} is in C'_ϕ . But then rsr^{-1} centralizes the image of ϕ , so that s centralizes the image of $\phi' = \text{Int}(r^{-1}) \circ \phi$. Thus, ϕ' is a cocharacter of L .

We claim that ϕ' is associated to u in L . Since the map $\varepsilon : \mathcal{N} \rightarrow \mathcal{U}$ is G -equivariant and thus restricts to a homeomorphism $\mathcal{N}(L) \rightarrow \mathcal{U}(L)$, we must see that ϕ' is associated to $\varepsilon^{-1}(u)$. Thus, we only must verify that ϕ' takes values in the derived group of L .

For each maximal torus S of $C_{\phi'}$, u is distinguished in $M = C_G(S)$ and the image of ϕ' lies in the derived group (M, M) (to see this last assertion, note that it holds for *some* such S since ϕ' is associated to u in G , and hence for all such S by conjugacy of maximal tori in $C_{\phi'}$).

We may choose a maximal torus $S \leq C_{\phi'}$ containing the connected center of L . Since $C_{\phi'}$ is normalized by s , we may also suppose by Proposition 18 that S is normalized by s . Then $C_S^o(s)$ is a torus in $C_L(u)$; since u is distinguished in L , we see that $C_S^o(s)$ is contained in (and hence coincides with) the connected center of L .

The subgroup M is normalized by s , and the proposition will follow if we can show that ϕ' takes values in the derived group of $C_M^o(s)$ (since $C_M^o(s) \subset L$).

We first claim that $C_S^o(s)$ is the maximal central torus of $C_M(s)$. Indeed, if $C_S^o(s) \subset S'$ with S' a central torus of $C_M(s)$, then S' centralizes s and u so that $S' \subset C_L(u)$; since $C_S^o(s)$ is the unique maximal torus of $C_L(u)$, $S' = C_S^o(s)$ as claimed.

The proposition is now a consequence of the lemma which follows. \square

Lemma 24. *Let M be a connected, reductive group, and suppose that σ is a semisimple automorphism of M . If S is a central torus in M and $C_S^o(\sigma)$ is the maximal central torus of $C_M(\sigma)$, then*

$$(3) \quad (C_M^o(\sigma), C_M^o(\sigma)) = C_{(M, M)}^o(\sigma).$$

Proof. The inclusion

$$(C_M^\circ(\sigma), C_M^\circ(\sigma)) \subseteq C_{(M,M)}^\circ(\sigma)$$

is immediate (by [Spr98, 2.2.8] the group on the left is connected; it is also evidently a σ -stable subgroup generated by commutators in M).

On the other hand, according to [Ste68, 9.4], $N = C_{(M,M)}^\circ(\sigma)$ is reductive. We claim that N is semisimple; if that is so then $N = (N, N)$; since $N \subseteq C_M^\circ(\sigma)$, equality in (3) will follow.

Write Z° for the connected center of M . Then $Z^\circ \cap (M, M)$ is finite, e.g. by [Spr98, 8.1.6]. Moreover, since $S \subseteq Z^\circ$, we see that $C_S^\circ(\sigma) \cap N$ is finite. Let R denote the radical of N (R is the maximal central torus of N). Since $C_M(\sigma) = C_{Z^\circ}(\sigma) \cdot N$, we see that R is a central torus in $C_M(\sigma)$. Since $C_S^\circ(\sigma)$ is by assumption the maximal central torus of $C_M(\sigma)$, we have that $R \subseteq C_S^\circ(\sigma) \cap N$ is finite, so $R = 1$ and N is indeed semisimple. \square

Remark 25. Though we shall not have occasion to use it here, the conclusion of Proposition 23 is true more generally: (*) if L is a pseudo-Levi subgroup, and if ϕ is a cocharacter of L associated to a unipotent $u \in L$, then ϕ is associated to u in G . This follows from Proposition 23 together with the observation that a Levi subgroup of L is a pseudo-Levi subgroup of G (Proposition 15(2)), and that (*) holds when L is a Levi subgroup.

8. ESTABLISHING THE MAIN RESULT

Let \mathcal{A} be the set of triples $\mathbf{a} = (L, tZ^\circ, u)$ where L is a pseudo-Levi subgroup of G with center Z , $tZ^\circ \in Z/Z^\circ$ satisfies $C_G^\circ(tZ^\circ) = L$, and $u \in L$ is a distinguished unipotent element. The action of G on itself by inner automorphisms determines an action of G on \mathcal{A} .

For $\mathbf{a} = (L, tZ^\circ, u) \in \mathcal{A}$, we set $u(\mathbf{a}) = u$, and we write $c(\mathbf{a}) \subset A(u)$ for the element $c(\mathbf{a}) = tC_G^\circ(u)$.

Let \mathcal{B} be the set of all pairs (u, x) where $u \in G$ is unipotent and $x \in A(u)$. The action of G on itself by inner automorphisms yields an action of G on \mathcal{B} .

To $\mathbf{a} \in \mathcal{A}$ we associate the pair $\Phi(\mathbf{a}) = (u(\mathbf{a}), c(\mathbf{a})) \in \mathcal{B}$.

Lemma 26. *Let $(u, c) \in \mathcal{B}$. Then there is $\mathbf{a} \in \mathcal{A}$ with $\Phi(\mathbf{a}) = (u, c)$.*

Proof. Choose a semisimple $t \in C_G(u)$ whose image in $A(u)$ represents c (Corollary 13). Let S be a maximal torus of $C_G^\circ(u, t)$. Then $L = C_G^\circ(t, S) = C_G^\circ(tS)$ is a pseudo-Levi subgroup of G containing u , and $L = C_G^\circ(tZ^\circ)$ where Z denotes the center of L (Proposition 15(2)).

It remains to show that $\mathbf{a} = (L, tZ^\circ, u)$ is in \mathcal{A} , i.e. that u is distinguished in L . Let A be a maximal torus of $C_L(u)$; we must show that A is in the center of L . Note that A is a subtorus of $B = C_G^\circ(u, t)$ and that A centralizes S . In particular, A is contained in the Cartan subgroup $H = C_B(S)$; by [Spr98, Prop 6.4.2] H is nilpotent and S is its unique maximal torus. Thus A is contained in S , hence A is central in L .

It is clear that $\Phi(\mathbf{a}) = (u, c)$; this completes the proof. \square

Lemma 27. *Let $\mathbf{a}, \mathbf{b} \in \mathcal{A}$, and suppose that $u = u(\mathbf{a}) = u(\mathbf{b})$. If $c(\mathbf{a})$ and $c(\mathbf{b})$ are conjugate in $A(u)$, then there is $g \in C_G(u)$ with $\mathbf{a} = g\mathbf{b}$.*

Proof. Write $\mathbf{a} = (L, tZ^o, u)$ and $\mathbf{b} = (L', t'Z'^o, u)$. By Proposition 15(3), we may choose the representatives t, t' such that $L = C_G^o(t)$ and $L' = C_G^o(t')$.

Let $\phi : k^\times \rightarrow L$ be a co-character associated to u for the pseudo-Levi subgroup L ; see Propositions 16 and 6. Then ϕ is associated to u in G as well; see Proposition 23. Evidently, $t \in C_\phi$, where C_ϕ is the Levi factor of $C_G(u)$ of Proposition 10. Similarly, t' lies in a Levi factor $C_{\phi'}$ of $C_G(u)$.

Consider the collection $\mathcal{M} = \{C_\phi \mid \phi \text{ is associated to } u\}$ of Levi factors of $C_G(u)$. Then any two Levi factors in \mathcal{M} are conjugate under $C_G^o(u)$ by Proposition 10. The previous paragraph shows that $t, t' \in \cup_{M \in \mathcal{M}} M$, hence we may apply Corollary 21. That corollary yields $g \in C_G(u)$ and a semisimple $s \in C_G^o(u, t)$, such that $gt'g^{-1} = ts$.

Choose a maximal torus S of $C_G^o(u, t)$ containing s . Then $S \subset L$ and S centralizes u ; since u is distinguished in L , it follows that $s \in S \subset Z^o$. We have

$$gL'g^{-1} = C_G^o(gt'g^{-1}) = C_G^o(ts).$$

Since $s \in Z^o$, we find that $L \subseteq C_G^o(ts)$. Thus $\dim L' \geq \dim L$. A symmetric argument shows that $\dim L' \leq \dim L$, hence equality holds. We deduce that $gL'g^{-1} = C_G^o(ts) = L$, and so $g\mathbf{b} = \mathbf{a}$ as desired. \square

Proof of Theorem 1. In the notation introduced in this section, Theorem 1 is equivalent to: Φ induces a bijection from the set \mathcal{A}/G of G -orbits on \mathcal{A} to the set \mathcal{B}/G of G -orbits on \mathcal{B} .

First note that $\Phi(g\mathbf{a}) = g\Phi(\mathbf{a})$ for each $\mathbf{a} \in \mathcal{A}$, so that indeed Φ induces a well-defined map $\bar{\Phi} : \mathcal{A}/G \rightarrow \mathcal{B}/G$. Lemma 26 implies that $\bar{\Phi}$ itself is surjective, hence also Φ is surjective. Finally, Lemma 27 shows that $\bar{\Phi}$ is injective; this proves the theorem. \square

9. CENTRALIZERS OF SEMISIMPLE ELEMENTS IN QUASISIMPLE GROUPS

In this section, we characterize the pseudo-Levi subgroups of G when the root system is irreducible (i.e. when G is quasisimple); the results are applied in the next section. The characterization we give is well-known (certainly in characteristic 0), but as we have not located an adequate reference (see Remark 31 below), and since the arguments are not too lengthy, we include most details.

Let T be any torus over k with co-character group Y (in the application, we take T to be a maximal torus of G). We denote by $V = Y \otimes \mathbf{R}$ the extension of Y to a real vector space, and by $\mathbf{T} = V/Y$ the resulting compact (topological) torus. If X is the character group of T , then X identifies naturally with the Pontrjagin dual $\hat{\mathbf{T}} = \text{Hom}(\mathbf{T}, \mathbf{R}/\mathbf{Z})$ of \mathbf{T} [note that we regard \mathbf{R}/\mathbf{Z} as a *multiplicative* group]. The following lemma due to T. A. Springer may be found in [Ste68, §5.1]

Lemma 28. (a) For each $t \in T$, there is $t' \in \mathbf{T}$ with the following property:

$$(*) \quad \text{for each } \lambda \in X, \lambda(t) = 1 \text{ if and only if } \lambda(t') = 1.$$

(b) Conversely, if $t' \in \mathbf{T}$ has finite order, relatively prime to p if $p > 0$, there is $t \in T$ for which $(*)$ holds.

Unless explicitly stated otherwise, we suppose in this section that R is irreducible, so that G is quasisimple.

Let $\tilde{S} = S \cup \{\alpha_0\}$ where $\alpha_0 = -\tilde{\alpha}$; thus \tilde{S} labels the vertices of the extended Dynkin diagram of the root system R . For any subset $J \subsetneq \tilde{S}$, let $R_J = \mathbf{Z}J \cap R$. Note that we do not require the characteristic to be good for G in this section.

Lemma 29. *Let T be our fixed maximal torus of G , and let \mathbf{T} be the corresponding compact topological torus. For $t \in \mathbf{T}$, put $R_t = \{\alpha \in R \mid \alpha(t) = 1\}$. Then there is $J \subsetneq \tilde{S}$ such that R_t is conjugate to R_J by an element of W , the Weyl group of R .*

Proof. Let $\tilde{t} \in V$ represent $t \in \mathbf{T}$. For some element \tilde{w} of the affine Weyl group $W_a = W \cdot \mathbf{Z}Y$, $\tilde{w}\tilde{t}$ lies in the fundamental alcove A_o in V (whose walls are labeled by \tilde{S}). The image of $\tilde{w}\tilde{t}$ in V is then wt , where w is the image of \tilde{w} in the finite Weyl group W , and $R_{wt} = w^{-1}R_t$. Thus, we suppose that t can be represented by a vector in A_o . In that case, let $J = \{\alpha \in \tilde{S} \mid \alpha(t) = 1\}$. Then the equality of R_t and R_J is proved in [Lus95, Lemma 5.4] (in *loc. cit.*, Lusztig works instead with the complex torus $Y \otimes \mathbf{C}/Y$, but his argument is readily adapted to the current situation). \square

For a subset $J \subsetneq \tilde{S}$, we consider the subgroup

$$L_J = \langle T, \mathcal{X}_\alpha \mid \alpha \in R_J \rangle.$$

Proposition 30. *Let $t \in G$ be semisimple. Then $C_G^o(t)$ is conjugate to a subgroup L_J for some $J \subsetneq \tilde{S}$.*

Proof. We may suppose that $t \in T$. Set $R_t = \{\alpha \in R \mid \alpha(t) = 1\}$. According to Lemma 14, $C_G^o(t)$ is generated by T and the \mathcal{X}_α with $\alpha \in R_t$. With notations as before, choose $t' \in \mathbf{T}$ with the property (*) of Lemma 28 for t . Thus $R_t = R_{t'}$. Lemma 29 implies that R_t and R_J are W -conjugate for some $J \subsetneq \tilde{S}$; thus $C_G^o(t)$ is conjugate in G to L_J as desired. \square

Remark 31. When k is an algebraic closure of a finite field, Proposition 30 was proved by D. I. Deriziotis, and is stated in [Hum95, 2.15]. See the last paragraph of *loc. cit.* §2.15 for a discussion.

In good characteristic, the converse of the previous proposition is true as well:

Proposition 32. *Suppose that the characteristic of k is good for G . Let $J \subsetneq \tilde{S}$, and let Z be the center of L_J . There is $t \in Z$ with $L_J = C_G^o(t)$.*

Proof. It suffices to suppose that G is adjoint. In that case, there are vectors $\varpi_\alpha^\vee \in Y$, $\alpha \in S$, dual to the basis S of X . We suppose that $\alpha_0 \in J$, since otherwise L_J is a Levi subgroup and the result holds (in all characteristics) e.g. by [Spr98, 6.4.3].

Denote by $\{\alpha_1, \dots, \alpha_r\} \subset S$ the simple roots which are not in J . Since $J \neq \tilde{S}$, we have $r \geq 1$. Write $\varpi_i^\vee = \varpi_{\alpha_i}^\vee$. Choose ℓ a prime number different from p , and let $s \in \mathbf{T}$ be the image of

$$\tilde{s} = \frac{\ell - (a_2 + \dots + a_r)}{a_1 \ell} \varpi_1^\vee + \frac{1}{\ell} \sum_{i=2}^r \varpi_i^\vee \in V.$$

We have written a_i for the coefficient a_{α_i} ; see eq. (2). If $r > 1$, the order of s is divisible by ℓ and divides $a_1 \ell$; if $J = \tilde{S} \setminus \{\alpha_1\}$, s has order a_1 . Since p is good, the order of s is thus relatively prime to p . If ℓ is chosen sufficiently large, we have $J = \{\beta \in \tilde{S} \mid \langle \beta, \tilde{s} \rangle \in \mathbf{Z}\}$. Since \tilde{s} lies in the fundamental alcove A_o , (the proof of) Lemma 29 implies that $R_s = R_J$. Choose an element $t \in T$ corresponding to $s \in \mathbf{T}$ as in Lemma 28(b). By Lemma 14, $C_G^o(t)$ is generated by T and the \mathcal{X}_α with $\alpha \in R_s$; thus $C_G^o(t) = L_J$ as desired. \square

10. EXPLICIT DESCRIPTIONS FOR SIMPLE AND ADJOINT G

In this section, we consider G simple of adjoint type. Thus the roots R span the weight lattice X over \mathbf{Z} and the root system is irreducible. The characteristic of k is assumed to be good for G throughout.

The results of the previous section show that in good characteristic, a pseudo-Levi subgroup in the sense of this paper (connected centralizer of a semisimple element) is the same as a pseudo-Levi subgroup in the sense of [Som98] (subgroup conjugate to some L_J).

Lemma 33. *Let L_J be a pseudo-Levi subgroup with center Z .*

- (1) *Put $d_J = \gcd(a_\alpha \mid \alpha \in \tilde{S} \setminus J)$. Then Z/Z° is cyclic of order d_J .*
- (2) *Every element of the character group of Z/Z° can be represented by a root in R .*

Proof. Since p is good, $\mathbf{Z}R/\mathbf{Z}J$ has no p -torsion. Thus the character group $X(Z/Z^\circ)$ is isomorphic to the torsion subgroup of $\mathbf{Z}R/\mathbf{Z}J$ as in [Som98, §2], so the proof of (1) in *loc. cit.* remains valid over k .

It is also true that $X(Z/Z^\circ)$ is naturally isomorphic to $\mathbf{Z}\bar{R}_J/\mathbf{Z}J$ where \bar{R}_J denotes the rational closure of R_J in R . We will show that the set \bar{R}_J surjects onto the latter cyclic group. Now \bar{R}_J is the root system of a Levi subgroup of G , and so it contains at most one irreducible component of type different than type A . Since the rank of $\mathbf{Z}R_J$ equals the rank of $\mathbf{Z}\bar{R}_J$, the type A components of \bar{R}_J play no role (every root sub-system is rationally closed in type A), and so we may assume that \bar{R}_J is irreducible. Then R_J is a root system with Dynkin diagram obtained by removing one simple root α from the extended Dynkin diagram of \bar{R}_J . Since there exists a positive root in \bar{R}_J whose coefficient on α is any number between 1 and d_J (note that d_J is necessarily the coefficient of the highest root of \bar{R}_J on α), (2) follows. \square

Lemma 34. *Let L be a pseudo-Levi subgroup with center Z .*

- (1) *For $t \in Z$, we have $L = C_G^\circ(tZ^\circ)$ if and only if tZ° generates Z/Z° .*
- (2) *If $u \in L$ is a distinguished unipotent element, then the group $N_G(L) \cap C_G(u)$ acts transitively on the generators of the cyclic group Z/Z° .*

Proof. We always have $L = C_G^\circ(Z) \subset C_G^\circ(tZ^\circ) \subset C_G^\circ(Z^\circ)$. Hence if tZ° generates Z/Z° , then clearly $L = C_G^\circ(tZ^\circ)$. For the converse, we may assume that $L = C_G^\circ(t)$ by Proposition 15. If tZ° fails to generate Z/Z° , then by the previous lemma there exists a root $\beta \in R$ such that $\beta(t) = 1$, but β is non-trivial on Z . By Lemma 14 this contradicts the fact that $C_G^\circ(Z) = C_G^\circ(t)$, and (1) follows.

Assertion (2) follows from [Som98, Prop. 8]. \square

Proposition 35. [Som98] *To a pair (L, u) of a pseudo-Levi subgroup L with center Z and distinguished unipotent element $u \in L$, assign the pair (u, c) where $c \in A(u)$ is the image of any generator of Z/Z° . Then this map defines a bijection between the G -orbits on the pairs (L, u) and the G -orbits on the pairs (u, c) .*

Proof. In view of Lemma 34, this follows from Theorem 1. \square

To determine the isomorphism type of the groups $A(u)$ we need to argue that the calculations in [Som98] remain valid over k .

Let \hat{G} be the group over \mathbf{C} with the same root datum as G . Since the characteristic is good, the Bala-Carter-Pommerening theorem shows that unipotent classes

of G and of \hat{G} are parametrized by their labeled diagram; cf. [Jan, 4.7 and 4.13]. It follows immediately that the G -orbits of pairs (L, u) as in the previous proposition are parametrized by the same combinatorial data as for \hat{G} ; namely, (L, u) corresponds to the pair (J, D_J) where J is a proper subset of \tilde{S} and L is conjugate to L_J (see Proposition 30), and where D_J is the labeled Dynkin diagram of the class of u in L . As in the remarks preceding [Som98, Remark 6], the G -orbit of (L, u) identifies with the W -orbit of (J, D_J) .

Now given a unipotent class in G with labeled diagram D , we are left with the task of determining which pairs (L, u) (up to G -conjugacy) as in the previous proposition are such that u has diagram D in G . Since an associated cocharacter of u in L is associated to u in G by Proposition 23, we may begin with the labeled diagram of u for L and produce by W -conjugation the labeled diagram of u for G ; see [Som98, §3.3]. It is now clear that our task is combinatorial: for a fixed $J \subsetneq \tilde{S}$, we must find all “distinguished” labeled diagrams for L_J which have D as a W -conjugate. The calculations are carried out in [Som98, §3.3, 3.4, 3.5], and they remain valid for k . Thanks to Proposition 35, this gives a bijection between the conjugacy classes of $A(u)$ and those of $A(\hat{u})$.

According to Lemma 33(1), the order of a representative element in $A(u)$ for the class determined by the pair (L, u) is independent of the ground field. According to [Som98, §3.4, §3.5], knowledge of the conjugacy classes and the orders of representing elements in $A(\hat{u})$ are sufficient to determine the group structure. The same then holds for $A(u)$, and we have proved:

Theorem 36. *For each unipotent element $u \in G$, let $\hat{u} \in \hat{G}$ be a unipotent element with the same labeled diagram as u . Then $A(u) \simeq A(\hat{u})$.*

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