NORMALITY OF VERY EVEN NILPOTENT VARIETIES IN D_{2l}

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ABSTRACT. For the classical groups, Kraft and Procesi [4], [5] have resolved the question of which nilpotent orbits have closures which are normal and which are not, with the exception of the very even orbits in D_{2l} which have partition of the form (a^{2k}, b^2) for $a > b$ even natural numbers satisfying $ak + b = 2l$.

In this article, these orbits are shown to have normal closure using the technique of [8].

1. INTRODUCTION

The question of which nilpotent orbits in a simple Lie algebra (defined over the complex numbers) have normal closure has been studied by Kostant, Hesselink, Kraft-Procesi, Broer, and others. The only open cases in the classical groups before this paper were a portion of the very even orbits in the special orthogonal groups. The very even orbits are those which are one orbit under an orthogonal group, but split into two orbits under the special orthogonal group. The elements in these orbits are characterized by the fact that, when viewed in the Lie algebra of all matrices, their Jordan blocks all have even size (additionally, there must be an even number of blocks of each even size). Kraft and Procesi [5] showed for the very even orbits that when the Jordan blocks have one (even) size, the orbit closure is normal and when these blocks are comprised of three or more distinct (even) sizes, the closure is not normal. They also showed that when there are two distinct (even) sizes and the smaller block size occurs with multiplicity at least four, then the closure is not normal.

In this paper, we resolve the remaining cases: namely, in type D_{2l} when the size of the Jordan blocks take the form (a^{2k}, b^2) where $a > b$ are even natural numbers satisfying $ak + b = 2l$. The main result is that these orbits do have normal closure. Our method is based upon that of [8] and uses the fact that the orbit corresponding to the partition $(a^{2k}, b+1, b-1)$ is known to have normal closure by results in [5]. Along the way we prove and use a new result about vector bundles on the flag variety in type D_{2l+1} (Theorem 4.1).

2. SOME LEMMAS IN A_l

We retain the notation of [8]. Throughout, G is a connected simple algebraic group over $\mathbf{C}$, B a Borel subgroup, T a maximal torus in B. The simple roots are denoted by Π, and they correspond to the Borel subgroup opposite to B. Let $\{\omega_i\}$ be the fundamental weights of G corresponding to Π. If $\alpha \in \Pi$, then P_α denotes the parabolic subgroup of semisimple rank one containing B and corresponding to α. If P is a parabolic subgroup of G, we denote by $\mathfrak{u}_P$ the Lie algebra of its unipotent radical.

We recall

Proposition 2.1. [3] *Let V be a rational representation of B and assume that V extends to a representation of the parabolic subgroup P_α where α is a simple root. Let $\lambda \in X^*(T)$ be such that $m = \langle \lambda, \alpha^\vee \rangle \geq -1$.*

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Then there is a G-module isomorphism

$$H^i(G/B, V\otimes\lambda) = H^{i+1}(G/B, V\otimes\lambda-(m+1)\alpha) \text{ for all } i\in\mathbb{Z}.$$

In particular, if $m=-1$, then all cohomology groups vanish.

For the rest of this section and the next, let $G = SL_{l+1}(\mathbf{C})$. We index the simple roots $\Pi = \{\alpha_j\}$ so that α_1 is an extremal root and α_j is next to α_{j+1} in the Dynkin diagram of type A_l.

The following lemma follows easily from several applications of the previous proposition.

Lemma 2.2. [7] *Let V be a rational representation of B which extends to a representation of P_{α_j} for $a\le j\le b$. Let $\lambda\in X^*(T)$ be such that $\langle\lambda,\alpha_j^\vee\rangle = 0$ for $a<j\le b$. Set $r = \langle\lambda,\alpha_a^\vee\rangle$ and assume that $a-b-1\le r\le -1$. Then $H^*(V\otimes\lambda)=0$.*

A similar statement holds by applying the non-trivial automorphism to the Dynkin diagram of type A_l. We use this lemma to prove

Lemma 2.3. *Let V be a representation of B which is stable under the parabolic subgroups P_{α_j} for $1\le j\le b$. Let $\lambda\in X^*(T)$ be such that $\langle\lambda,\alpha_a^\vee\rangle = 1$ for some a satisfying $1\le a<b$. Assume that $\langle\lambda,\alpha_j^\vee\rangle=0$ for $1\le j\ne a<b$. Set $k=\langle\lambda,\alpha_b^\vee\rangle$. If $-b-1\le k\le -1$ and $k+b-a\ne -1$, then $H^*(V\otimes\lambda)=0$.*

Proof. If $k+b-a\ge 0$, the result follows directly from Lemma 2.2. On the other hand, if $k+b-a\le -2$, then as in the proof of Lemma 2.2 in [7],

$$H^i(V\otimes\lambda) = H^{i+b-a}(V\otimes\mu)$$

where

$$\mu = \lambda + (-k-1)\alpha_b + (-k-2)\alpha_{b-1} + \cdots + (-k-b+a)\alpha_{a+1}.$$

Now $\langle\mu,\alpha_j^\vee\rangle = 0$ for $1\le j<a$ and $\langle\mu,\alpha_a^\vee\rangle = k+b-a+1$. By the hypothesis on λ and the present assumption about $k+b-a$, we have

$$-a\le k+b-a+1\le -1.$$

Then Lemma 2.2 yields the desired vanishing. □

3. A THEOREM FOR A_l (REVIEW)

Let P_m denote the maximal proper parabolic subgroup of $G=SL_{l+1}(\mathbf{C})$ containing B corresponding to all the simple roots except α_m. Denote the Lie algebra of the unipotent radical of P_m by $\mathfrak{u}_m$. The action of P_m on $\mathfrak{u}_m$ gives a representation of P_m (and also B). Denote the dual representation by $\mathfrak{u}_m^*$. Set $m'=\min\{m,\ l+1-m\}$. In [7], Lemma 2.2 and Proposition 2.1 were used to prove

Theorem 3.1. [7] *Let r be an integer in the range $2m'-2-l\le r\le 0$. Then there is a G-module isomorphism*

$$H^i(G/B, S^n\mathfrak{u}_m^*\otimes r\omega_m) = H^i(G/B, S^{n+rm'}\mathfrak{u}_{l+1-m}^*\otimes -r\omega_{l+1-m}) \text{ for all } i,n\ge 0.$$

4. A THEOREM FOR D_{2l+1}

Theorem 3.1 has an analog in type D_{2l+1}. We label the simple roots of G of type D_{2l+1} as in [6], so α_{2l-1} lies at the branched vertex of the Dynkin diagram. Let P be the maximal proper parabolic subgroup containing B corresponding to all the simple roots except α_{2l}. And let P' be the maximal proper parabolic subgroup containing B corresponding to all the simple roots except α_{2l+1} (so P and P' are interchanged by an outer automorphism of G).

Theorem 4.1. *Let r be an integer in the range $-3 \leq r \leq 0$. Then there is a G-module isomorphism*

$$H^i(G/B, S^n\mathfrak{u}_P^* \otimes r\omega_{2l}) = H^i(G/B, S^{n+rl}\mathfrak{u}_{P'}^* \otimes -r\omega_{2l+1}) \text{ for all } i, n \geq 0.$$

Proof. **Step 1.** In this step, r may be an arbitrary integer. Consider the intersection $V = \mathfrak{u}_P \cap \mathfrak{u}_{P'}$. We will show in Step 1 that for all i, n

$$H^i(S^n\mathfrak{u}_P^* \otimes r\omega_{2l}) = H^i(S^nV^* \otimes r\omega_{2l}). \tag{1}$$

We begin by taking the Koszul resolution of the short exact sequence

$$0 \to U \to \mathfrak{u}_P^* \to V^* \to 0$$

(this defines U) and tensoring it with $r\omega_{2l}$. This gives

$$0 \to \cdots \to S^{n-j}\mathfrak{u}_P^* \otimes \wedge^jU \otimes r\omega_{2l} \to \cdots \to S^n\mathfrak{u}_P^* \otimes r\omega_{2l} \to S^nV^* \otimes r\omega_{2l} \to 0.$$

We claim that $H^*(S^{n-j}\mathfrak{u}_P^* \otimes \wedge^jU \otimes r\omega_{2l}) = 0$ for $1 \leq j \leq \dim U$ from which Equation 1 will follow. The T-weights of U are those of the form $\alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l}$, where $1 \leq k \leq 2l$. Therefore, if λ is a T-weight of $\wedge^jU$, then λ is of the form

$$(0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2, \ldots, j-1, \ldots, j-1, j, \ldots, j, 0)$$

in the basis of simple roots. If this expression contains a subsequence of the form $m, m, m+1$, then λ will have inner product -1 with the simple coroot corresponding to the middle m. Hence $H^*(Q \otimes \lambda) = 0$ where Q is any P-representation by Proposition 2.1. The same result holds if there are any 0's in the initial part of the expression. Therefore, we are reduced to considering those λ of the form

$$(1, 2, 3, \ldots, j-1, j, j, \ldots, j, 0).$$

Such a λ satisfies $\langle \lambda, \alpha_{2l+1}^\vee \rangle = -j$ with the exception of the case $j = 2l$, where instead $\langle \lambda, \alpha_{2l+1}^\vee \rangle = -j + 1 = -2l + 1$. In the latter case $H^*(Q \otimes \lambda) = 0$ by Lemma 2.2 applied to to the parabolic subgroup with Levi factor of type A_{2l} consisting of all simple roots except α_{2l}. For the cases where $j < 2l$, we can apply Lemma 2.3, also for the A_{2l} consisting of all simple roots except α_{2l}. In that case, $a = j$, $b = 2l$, $k = -j$ and so $k + b - a = 2l - 2j$, which, being an even number, is never -1. Also, clearly $-b - 1 \leq k \leq -1$. Thus we conclude that for all weights λ appearing in $\wedge^jU$, we have $H^*(Q \otimes \lambda) = 0$ for any P-representation Q. Hence for $Q := S^{n-j}\mathfrak{u}_P^* \otimes r\omega_{2l}$, it follows that $H^*(Q \otimes \wedge^jU) = 0$ by the usual filtration argument.

Step 2. Let V_1 be the B-stable subspace of $\mathfrak{u}$ consisting of the direct sum of all root spaces $\mathfrak{g}_\alpha$ where $-\alpha$ is bigger than or equal to the root

$$(0, \ldots, 0, 1, 2, 1, 1)$$

in the usual partial ordering on roots. Let V_2 be the B-stable subspace of $\mathfrak{u}$ consisting of the direct sum of all root spaces $\mathfrak{g}_\alpha$ where $-\alpha$ is bigger than or equal to the root

$$(0, 0, \ldots, 0, 1, 2, 2, 1, 1).$$

Let μ be a weight of the form $r\,\omega_{2l} + s\,\omega_{2l+1}$ where r, s are integers. Assume that $-3 \leq r \leq -1$ and that $s = 0$ if $r = -3$. In this step we show for all $n \geq 0$ that

$$H^*(S^n V_1^* \otimes \mu) = 0. \tag{2}$$

Take the Koszul resolution of

$$0 \to U_2 \to V_1^* \to V_2^* \to 0$$

(this defines U_2) and tensor it with μ. We will show that

$$H^*(S^n V_2^* \otimes \mu) = 0$$

and

$$H^*(S^{n-j} V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0$$

for $1 \leq j \leq 2l-2$ and then Equation (2) will follow (the dimension of U_2 is $2l-2$ as shown below).

The subspace V_2^* is stable under the minimal parabolic subgroups P_{α_m} for $m = 2l-1, 2l$, and $2l+1$. It follows from the assumption on μ that $H^*(S^n V_2^* \otimes \mu) = 0$ by Lemma 2.2 applied to the A_3 determined by the simple roots α_m for $m = 2l-1, 2l$, and $2l+1$.

Now the T-weights of U_2 are

$$\alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l-2} + 2\alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}$$

where $1 \leq k \leq 2l-2$. If λ is a weight of $\wedge^j U_2$, then λ is of the form

$$(0, \ldots, 0, 1, \ldots, 1, 2, \ldots, 2, \ldots, j-1, \ldots, j-1, j, \ldots, j, 2j, j, j)$$

in the basis of simple roots. As in the previous step, if there are any 0's present or if any of the integers between 1 and $j-1$ inclusive are repeated, then

$$H^*(Q \otimes \lambda) = 0$$

where $Q := S^{n-j} V_1^* \otimes \mu$ since Q is stable under the action of the parabolic subgroups P_{α_k} for $1 \leq k \leq 2l-2$. Hence we are reduced to considering those λ of the form

$$(1, 2, 3, \ldots, j-2, j-1, j, \ldots, j, 2j, j, j)$$

for $1 \leq j \leq 2l-2$. Such a λ satisfies $\langle \lambda, \alpha_{2l-2}^\vee \rangle = -j$ with the exception of $j = 2l-2$ where $\langle \lambda, \alpha_{2l-2}^\vee \rangle = -2l+3$. In the latter case $H^*(Q \otimes \lambda) = 0$ by Lemma 2.2 applied to the A_{2l-2} consisting of the first $2l-2$ simple roots. For the cases where $j < 2l-2$, we can apply Lemma 2.3, also for the A_{2l-2} consisting of the first $2l-2$ simple roots. In that case, $a = j$, $b = 2l-2$, $k = -j$ and so $k + b - a = 2l - 2j - 2$, which is never -1. Also, clearly $-b-1 \leq k \leq -1$. We therefore also have $H^*(Q \otimes \lambda) = 0$.

Consequently, if we filter $\wedge^j U_2$ by B-submodules such that the quotients are one-dimensional, we deduce that

$$H^*(S^{n-j} V_1^* \otimes \wedge^j U_2 \otimes \mu) = 0$$

for $1 \leq j \leq 2l-2$. Hence Equation (2) follows.

Step 3. In this step, we show that for all i, n

$$H^i(S^n V^* \otimes \mu) = H^i(S^{n-l} V^* \otimes \mu + \omega_{2l} + \omega_{2l+1}) \tag{3}$$

for μ as in Step 2.

We take the Koszul resolution of the short exact sequence

$$0 \to U_1 \to V^* \to V_1^* \to 0$$

(this defines U_1) and tensor it with μ arriving at

(4) $$0 \to S^{n-2l+1}V^* \otimes \wedge^{2l-1}U_1 \otimes \mu \to \cdots \to S^{n-j}V^* \otimes \wedge^j U_1 \otimes \mu \to \cdots \to S^n V^* \otimes \mu \to S^n V_1^* \otimes \mu \to 0$$

We first show that $H^*(S^{n-j}V^* \otimes \mu \otimes \lambda) = 0$ for any λ appearing in $\wedge^j U_1$ for $j \neq 0, l$. The weights of U_1 are

$$\alpha_k + \alpha_{k+1} + \cdots + \alpha_{2l-1} + \alpha_{2l} + \alpha_{2l+1}$$

where $1 \leq k \leq 2l-1$. If λ is a weight of $\wedge^j U_1$, then as in the previous steps we are quickly reduced to those λ of the form

$$(1, 2, 3, \ldots, j-2, j-1, j, \ldots, j, j, j, j)$$

for $1 \leq j \leq 2l-1$. Such a λ satisfies $\langle \lambda, \alpha_{2l-1}^\vee \rangle = -j$ with the exception of $j = 2l-1$ where $\langle \lambda, \alpha_{2l-2}^\vee \rangle = -2l+2$. The latter vanishing follows from Lemma 2.2 applied to to the A_{2l-1} consisting of the first $2l-1$ simple roots. For the cases where $j < 2l-1$, we can apply Lemma 2.3, also for the A_{2l-1} consisting of the first $2l-1$ simple roots. In that case, $a = j$, $b = 2l-1$, $k = -j$ and so $k + b - a = 2l - 2j - 1$, which is -1 only when $j = l$. Therefore, we deduce that

$$H^*(S^{n-j}V^* \otimes \wedge^j U_1 \otimes \mu) = 0$$

when $j \neq 0, l$. Furthermore,

$$H^i(S^{n-l}V^* \otimes \wedge^l U_1 \otimes \mu) = H^i(S^{n-l}V^* \otimes \lambda \otimes \mu),$$

where $\lambda = (1, 2, 3, \ldots, l-1, l, \ldots, l, l, l)$. Now $S^{n-l}V^* \otimes \mu$ is stable under P_{α_t} for $1 \leq t \leq 2l-1$. Hence $l-1$ applications of Proposition 2.1 for t in the range $2l-1 \geq t \geq l+1$ yields

$$H^i(S^{n-l}V^* \otimes \lambda \otimes \mu) = H^{i+l-1}(S^{n-l}V^* \otimes \mu + \omega_{2l} + \omega_{2l+1}).$$

By breaking Equation (4) into short exact sequences and taking cohomology on G/B, we conclude that

$$H^i(S^n V^* \otimes \mu) = H^i(S^{n-l}V^* \otimes \mu + \omega_{2l} + \omega_{2l+1}),$$

where we are using

$$H^*(S^n V_1^* \otimes \mu) = 0$$

from Step 2.

Step 4. We obtain the theorem by using Step 3 repeatedly, starting with $\mu = r\omega_m$ with r in the prescribed range of the statement of the theorem. After $-r$ steps we arrive at

$$H^i(S^n V^* \otimes r\omega_{2l}) = H^i(S^{n+rl}V^* \otimes -r\omega_{2l+1}),$$

for all i, n. The proof is completed by using Step 1 and the symmetric version of Equation 1 (obtained by applying an outer automorphism of G) which gives

$$H^i(S^{n+rl}V^* \otimes -r\omega_{2l+1}) = H^i(S^{n+rl}\mathfrak{u}_{P'}^* \otimes -r\omega_{2l+1})$$

for all i, n.

□

In what follows, we will use Theorem 3.1 in the more general situation of Section 4 in [8]. Similarly we can apply Theorem 4.1 in an analogous general situation. Namely, suppose G is of general type and P is a parabolic subgroup of G containing B with Levi factor L containing a simple factor of type A_{2l}. Furthermore, suppose this simple factor belongs to a Levi subgroup L' of G of type D_{2l+1} and $[L, L'] \subset L'$. Then the analog in G of Theorem 4.1 holds just as the analog of Theorem 3.1 does in Proposition 6 in [8].

5. Main theorem

For the rest of the paper G is connected of type D_{2l}. We want to show that both nilpotent orbits in $\mathfrak{g}$ with partition (a^{2k}, b^2) for $a > b$ even natural numbers with $ak + b = 2l$ have normal closure. Let $\mathcal{O}$ denote one of these two orbits.

Following the idea of [8], we find a nilpotent orbit $\mathcal{O}'$ which we already know has normal closure and which contains $\mathcal{O}$ in its closure. If we can show that the regular functions on $\mathcal{O}$ are naturally a quotient of the regular functions on $\mathcal{O}'$, then it follows that $\mathcal{O}$ also has normal closure. To that end we consider the nilpotent orbit $\mathcal{O}'$ in $\mathfrak{g}$ with partition $\lambda = (a^{2k}, b + 1, b - 1)$.

Lemma 5.1. *The closure of $\mathcal{O}'$ is normal.*

Proof. The only minimal degenerations of $\mathcal{O}'$ in $\mathfrak{g}$ are the two orbits with partition $\mu = (a^{2k}, b^2)$ (which together are one orbit for the full orthogonal group of rank $2l$). Hence by [5] the singularity of the closure of $\mathcal{O}'$ along the union of these two orbits is smoothly equivalent to the singularity of the closure of the orbit with partition (2) along the orbit with partition $(1, 1)$ in type A_1 (we remove the first $2k$ rows from λ and μ, and then remove the first $b - 1$ columns from the resulting partitions). Hence this is a singularity of type A_1 and so by [5], $\mathcal{O}'$ has normal closure. ☐

Lemma 5.2. *The orbit $\mathcal{O}'$ is a Richardson orbit for any parabolic with Levi factor of type*

$$\overbrace{A_{2k-1} \times \cdots \times A_{2k-1}}^{\frac{a-b}{2}-1} \times A_{2k} \times A_{2k} \times \overbrace{A_{2k+1} \times \cdots \times A_{2k+1}}^{\frac{b}{2}-1}.$$

Any parabolic with Levi factor of type

$$\overbrace{A_{2k-1} \times \cdots \times A_{2k-1}}^{\frac{a-b}{2}} \times \overbrace{A_{2k+1} \times \cdots \times A_{2k+1}}^{\frac{b}{2}}$$

has Richardson orbit one or the other of the two nilpotent orbits with partition (a^{2k}, b^2).

Proof. Both statements follow from Section 7 in [2]. ☐

It will be convenient to represent parabolic subgroups containing B by the simple roots of G which are **not** simple roots of their Levi factors. Thus we can speak of such a parabolic subgroup as a subset of the numbers 1 to $2l + 1$, with each number i corresponding to the simple root α_i.

Set $d = a - b$ and let P' be the parabolic represented by

$$\{2k + 1, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l\}$$

and let P'' be represented by

$$\{2k + 1, 4k + 2, 6k + 2, \ldots, kd + 2, k(d + 2) + 2, k(d + 4) + 4, k(d + 6) + 6, \ldots, 2l - 2k - 2, 2l - 1\},$$

so P' are P'' are interchanged by an outer automorphism of D_{2l+1}. By the previous lemma $\mathcal{O}'$ is Richardson for both P' and P''. Let P be the parabolic represented by

$$\{2k, 4k+2, 6k+2, \dots, kd+2, k(d+2)+2, k(d+4)+4, k(d+6)+6, \dots, 2l-2k-2, 2l\}.$$

Then without loss of generality we can take $\mathcal{O}$ to be the Richardson orbit for P (again by the previous lemma).

Theorem 5.3. *There is a short exact sequence for all* $n \geq 0$

$$0 \to H^0(S^{n-2l-k(a-4)+1}\mathfrak{u}^*_{P''} \otimes \nu) \to H^0(S^n\mathfrak{u}^*_{P'}) \to H^0(S^n\mathfrak{u}^*_P) \to 0, \tag{5}$$

where $\nu = \omega_{4k+2}$ *if* $a > 4$ *and* $\nu = 2\omega_{2l-1}$ *if* $a = 4$ *(and hence* $b = 2$*).*

Proof. We use two elements from the proof of Theorem 3.1 in [7]. Let P_1 be the parabolic represented by

$$\{2k+2, 4k+2, 6k+2, \dots, kd+2, k(d+2)+2, k(d+4)+4, k(d+6)+6, \dots, 2l-2k-2, 2l\}$$

and set $V = \mathfrak{u}_P \cap \mathfrak{u}_{P_1}$. Then Step 1 of the proof of Theorem 3.1 (for a group of type A_{4k+1} applied to the first $4k+1$ simple roots of G) yields the isomorphism $H^i(S^n\mathfrak{u}^*_P) = H^i(S^nV^*)$ for all i, n. And Step 3 of the proof Theorem 3.1 yields the long exact sequence

$$\cdots \to H^i(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) \to H^i(S^n\mathfrak{u}^*_{P'}) \to H^i(S^nV^*) \to H^{i+1}(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) \to \dots \tag{6}$$

where μ equals

$$(1, 2, 3, \dots, 2k, 2k+1, 2k, \dots, 2, 1, \overbrace{0, 0, \dots, 0}^{2l-4k-1}).$$

This is obtained by taking the Koszul resolution of

$$0 \to U \to \mathfrak{u}^*_{P'} \to V^* \to 0$$

(this defines U) and simplifying the terms.

The remainder of the proof involves showing that

$$H^i(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) = H^i(S^{n-2l-k(a-4)+1}\mathfrak{u}^*_{P''} \otimes \nu)$$

for all i, n.

This is carried out by using Theorem 3.1 numerous times (for $r = -1$ and the l in that theorem equal to either $4k$ or $4k+1$ and $m' = 2k$ or $2k+1$, respectively) and Theorem 4.1 once (for $r = -2$ and the l in that theorem equal to k).

After $\frac{a-b-2}{2}$ applications of Theorem 3.1 with $r = -1$, and the l there equal to $4k$ and $m' = 2k$, we have

$$H^i(S^{n-2k-1}\mathfrak{u}^*_{P'} \otimes \mu) = H^i(S^{n-k(a-b)-1}Q^*_1 \otimes \mu_1)$$

where μ_1 equals

$$(1, 2, 3, \dots, 2k, \overbrace{2k+1, \dots, 2k+1}^{k(a-b-2)+1}, 2k, \dots, 2, 1, \overbrace{0, 0, \dots, 0}^{k(b-2)+b-1}),$$

and Q_1 is the Lie algebra of the unipotent radical of

$$\{2k+1, 4k+1, 6k+1, \dots, k(d-2)+1, kd+1, k(d+2)+2, k(d+4)+4, k(d+6)+6, \dots, 2l-2k-2, 2l\}.$$

Next, we apply Theorem 3.1 $\frac{b-2}{2}$ more times with $r = -1$, and the l there equal to $4k+1$ and $m' = 2k+1$, to obtain

$$H^i(S^{n-k(a-b)-1}Q_1^* \otimes \mu_1) = H^i(S^{n-ka+2k-b/2}Q_2^* \otimes \mu_2)$$

where μ_2 equals

$$(1, 2, 3, \ldots, 2k, \overbrace{2k+1, \ldots, 2k+1}^{2l-4k-1}, 2k, \ldots, 2, 1, 0),$$

and Q_2 is the Lie algebra of the unipotent radical of

$$\{2k+1, 4k+1, 6k+1, \ldots, k(d-2)+1, kd+1, k(d+2)+3, k(d+4)+5, k(d+6)+7, \ldots, 2l-2k-1, 2l\}.$$

Next, we use Theorem 4.1 with $r = -2$ for the case D_{2k+1} applied to the simple roots α_i of G with $2l-2k \leq i \leq 2l$. This yields

$$H^i(S^{n-ka+2k-b/2}Q_2^* \otimes \mu_2) = H^i(S^{n-ka-b/2}Q_3^* \otimes \mu_3)$$

where μ_3 equals

$$(1, 2, 3, \ldots, 2k, \overbrace{2k+1, \ldots, 2k+1}^{2l-4k-1}, 2k+2, 2k+3, 2k+4, \ldots, 4k, 2k+1, 2k),$$

and Q_3 is the Lie algebra of the unipotent radical of

$$\{2k+1, 4k+1, 6k+1, \ldots, k(d-2)+1, kd+1, k(d+2)+3, k(d+4)+5, k(d+6)+7, \ldots, 2l-2k-1, 2l-1\}.$$

If $2l-4k-1 = 1$, which is the case if and only if $a = 4$ and $b = 2$, we have $\mu_3 = 2\omega_{2l-1}$ and the parabolic in question is $\{2k+1, 2l-1\}$, which is just P''.

On the other hand, if $a > 4$, we continue by using Theorem 3.1 another $\frac{b-2}{2}$ times followed by another $\frac{a-b-2}{2}$ times (in reverse of how we have just used it). The result is that

$$H^i(S^{n-ka-b/2}Q_3^* \otimes \mu_3) = H^i(S^{n-2ka+4k-b+1}Q_4^* \otimes \mu_4)$$

where μ_4 equals

$$(1, 2, 3, \ldots, 4k+1, \overbrace{4k+2, \ldots, 4k+2}^{2l-4k-3}, 2k+1, 2k+1),$$

and Q_4 is the Lie algebra of the unipotent radical of

$$\{2k+1, 4k+2, 6k+2, \ldots, k(d-2)+2, kd+2, k(d+2)+2, k(d+4)+4, k(d+6)+6, \ldots, 2l-2k-2, 2l-1\}.$$

The latter parabolic is exactly P'' and $\mu_4 = \omega_{4k+2}$. Furthermore, $n-2ka+4k-b+1 = n-2l-ka+4k+1$ since $ak + b = 2l$.

Hence when $a = 4$ or $a > 4$, we have shown that

$$H^i(S^{n-2k-1}\mathfrak{u}_{P'}^* \otimes \mu) = H^i(S^{n-2l-k(a-4)+1}\mathfrak{u}_{P''}^* \otimes \nu)$$

for all i, n. We finish the proof by observing that ν extends to a character of P'' and it is dominant. Hence $H^i(S^{n-2l-k(a-4)+1}\mathfrak{u}_{P''}^* \otimes \nu) = 0$ for $i > 0$ as in [1]. Similarly, $H^i(S^n\mathfrak{u}_{P'}^*) = 0$ and $H^i(S^n\mathfrak{u}_P^*) = 0$ for $i > 0$ and the proof is complete using Equation (6). □

Corollary 5.4. *The closure of $\mathcal{O}$ is normal.*

Proof. The functions of degree n on $\mathcal{O}'$ (and also its closure since the closure is normal) as a G-module are naturally isomorphic to $H^0(S^n\mathfrak{u}_{P'}^*)$. Also the functions of degree n on $\mathcal{O}$ as a G-module are naturally isomorphic to $H^0(S^n\mathfrak{u}_P^*)$. This follows since both $\mathcal{O}'$ and $\mathcal{O}$ have trivial G-equivariant fundamental group when G is adjoint (see [2]). Thus the short exact sequence of the theorem and Lemma 5.1, together with the discussion in Section 3 of [8], yield the result. □

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