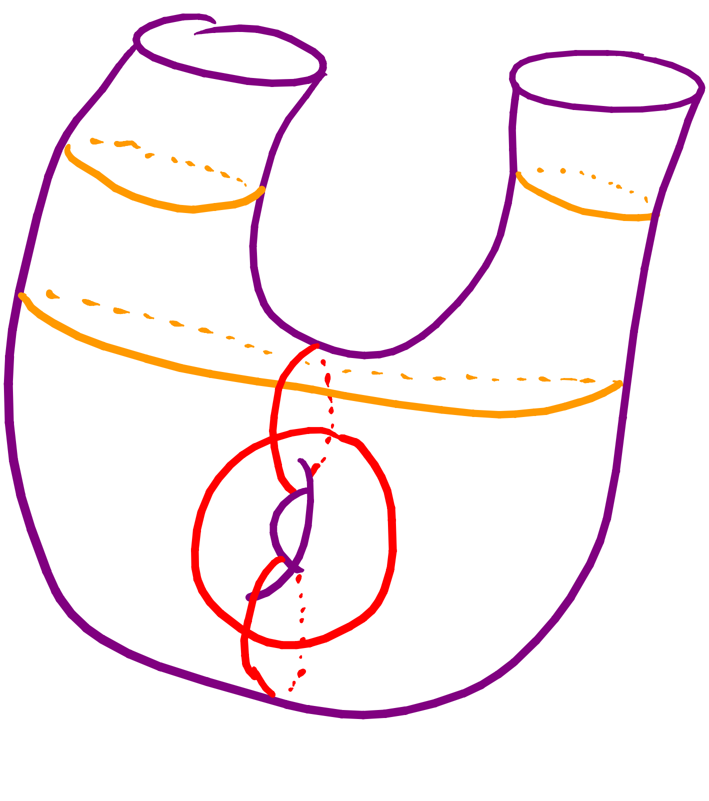
Monoids the Mapping Class Group and Contact Geometry



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Giroux Correspondence

$\left\{\begin{matrix}\text{contact structures} \\ \text{up to isotopy}\end{matrix}\right\} \underset{\text{1-1 corresp.}}{\Longleftrightarrow} \left\{\begin{matrix}\text{open book decomp.} \\ \text{up to positive stab.}\end{matrix}\right\}$

- This result has been a key to recent advances in contact geom.
- It has also been a major factor in many recent applications of contact geometry to low-dimensional topology.

  e.g.
  - Kronheimer and Mrowka's proof that all non-trivial knots satisfy property P
  - Ozsváth and Szabó's surgery characterization of 
- In this talk we will survey results relating contact geometry and topology to the mapping class group.

Contact Structures

Recall a contact structure is an (oriented) plane field ξ on a 3-manifold M that is "maximally non-integrable".

i.e. $\exists$ a 1-form α such that

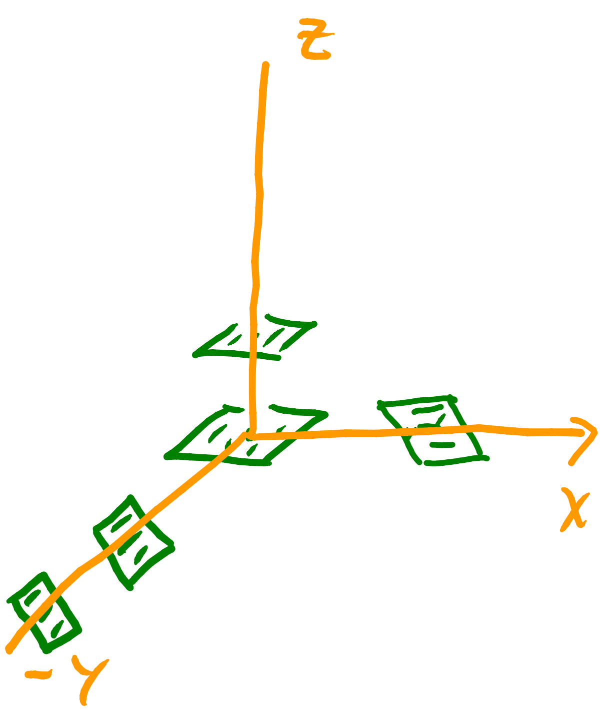
$$\boxed{\begin{array}{l} \xi = \ker \alpha \\ \alpha \wedge d\alpha > 0 \end{array}}$$

example:

1) on $\mathbb{R}^3$, let $\alpha = dz + r^2 d\theta$.

and $\xi_{std} = \ker \alpha$

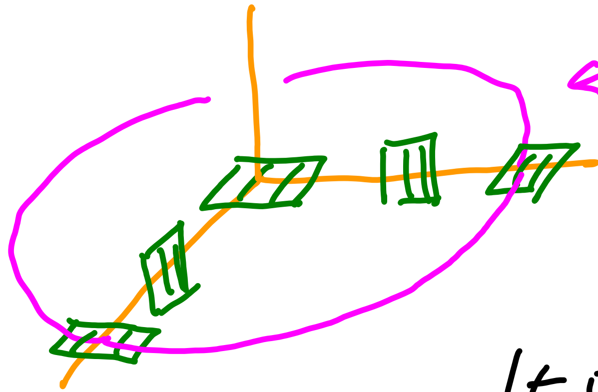
$= \text{span}\{\frac{\partial}{\partial r}, r^2 \frac{\partial}{\partial z} - \frac{\partial}{\partial \theta}\}$



Darboux:

all contact structures look locally like this one

2) on $\mathbb{R}^3$, let $\alpha = \cos r\, dz + r \sin r\, d\theta$ and $\xi_{ot} = \ker \alpha$



$D = \{(r,\theta,z) : r \leq \pi,\ z = 0\}$ is tangent to ξ_{ot} along the boundary.

It is called an overtwisted disk

- A contact manifold (M, ξ) is called overtwisted if it contains such a disk and otherwise it is called tight
- Clearly $(\mathbb{R}^3, \xi_{ot})$ is overtwisted
- Bennequin proved $(\mathbb{R}^3, \xi_{std})$ is tight
- Eliashberg classified overtwisted contact structures (just about algebraic topology)
- Tight contact structures are much more subtle! Used in applications mentioned above.

Open Book Decompositions

An open book decomposition of a 3-manifold (closed, oriented) M is a pair (L, π) where

1) L is a link in M called the binding and
2) $\pi : (M - L) \to S^1$ is a fibration so that $\Sigma_\theta = \overline{\pi^{-1}(\theta)}$ is a Seifert surface for the link L

the Σ_θ are called pages

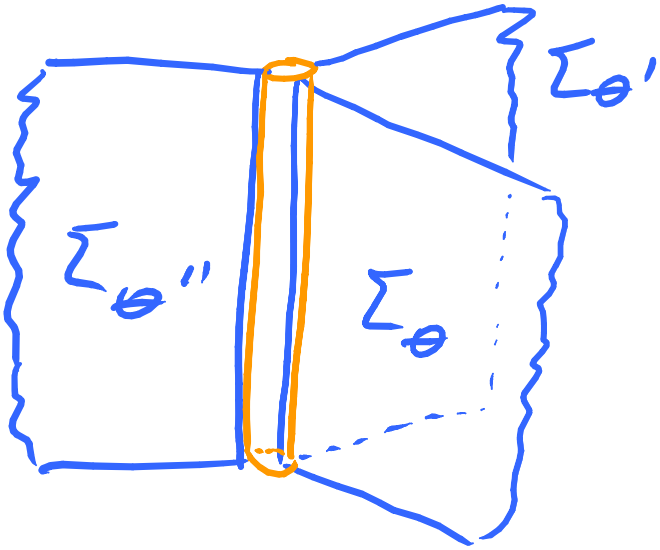
Examples:

1) let U be the unknot in S^3

$$(S^3 - U) = (\mathbb{R}^3 - z\text{-axis}) \qquad (r, \theta, z)$$

$$\downarrow \pi \qquad\qquad \downarrow$$

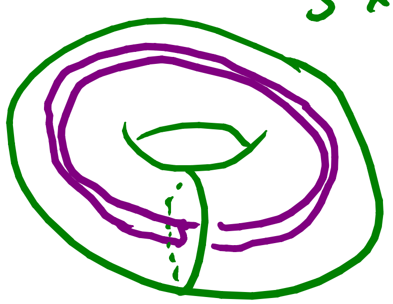
$$S^1 \qquad\qquad \theta$$



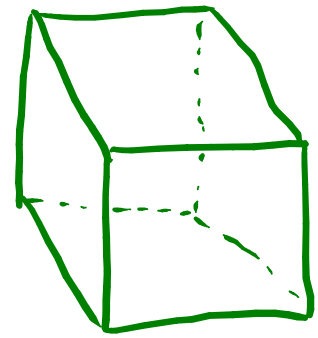
so (U,π) is an open book of S^3

2) let H_+ be the Hopf link

note $(S^3 - H_+) = \underbrace{(S^3 - U)}_{S^1\times\mathbb{R}^2} - \underset{\substack{\uparrow \\ S^1\times\{pt\}}}{U'}$

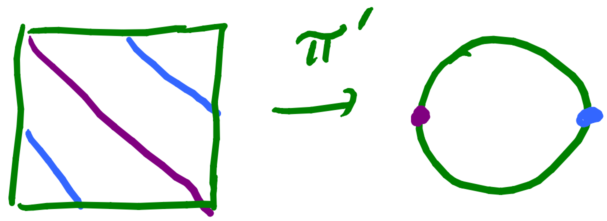
$=$  $= T^2\times(0,1)$

$T^2 \times (0,1) =$



identify top to bottom and front to back

let $\pi' : T^2 \to S^1$



$$\pi : T^2 \times [0,1] \to S^1 : (p,t) \mapsto \pi'(p)$$

this fibers $S^3 - H_+$ with fiber

alternately, let $S^3 \subseteq \mathbb{C}^2$ be the unit sphere

$$H_+ = \{(z_1, z_2) \in S^3 \mid z_1 z_2 = 0\}$$

$$\pi : (S^3 - H_+) \to S^1 : (z_1, z_2) \mapsto \frac{z_1 z_2}{|z_1 z_2|}$$

3) If $p : \mathbb{C}^2 \to \mathbb{C}$ is any polynomial that vanishes at $(0,0)$ and has no critical points (except possibly $(0,0)$) inside S^3, then

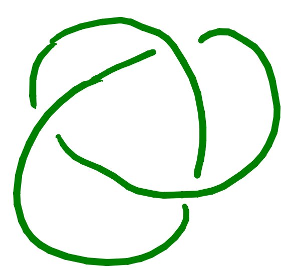
$$K_p = \{(z_1, z_2) \in S^3 : p(z_1, z_2) = 0\}$$

is fibered with fibration

$$\pi_p : (S^3 - K_p) \to S^1 : (z_1, z_2) \mapsto \frac{f(z_1, z_2)}{|f(z_1, z_2)|}$$

so (K_p, π_p) is an open book of S^3

exercise: find a polynomial so that K_p is



$Th^{\underline{m}}$ (Alexander): Every 3-manifold has an open book decomposition.

In 2000 Giroux made the following definition: a contact structure ξ on M is <u>supported</u> by an open book (L, π) if there is a 1-form α s.t.

1) $\xi = \ker \alpha$
2) $\alpha(v) > 0$ $\forall v$ pos tangent to L
3) $\pi^*(d\theta) \wedge d\alpha > 0$ where θ is the coord on S^1

(i.e. $d\alpha$ is an area form on each page)

<u>example:</u>

1) (U, π) supports the standard contact structure on S^3
2) (H_+, π) does too

<u>Thm (Thurston-Winkelnkemper 1975)</u>

$$\boxed{\text{Every open book supports a contact structure}}$$

Remark 1) This shows all three manifolds have contact structures

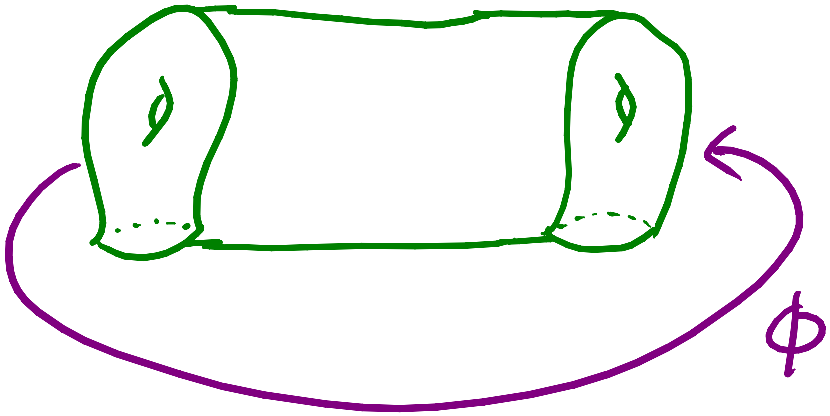
2) Not hard to show supported contact str is unique

To "prove" this theorem we need to reinterperate open books from the monodromy perspective

Given an open book (L, π) of M let $\Sigma = \overline{\pi^{-1}(\theta)}$ for some $\theta \in S^1$

note: $$\begin{array}{ccc} ((M-L) \setminus \Sigma) & & \Sigma \times [0,1] \\ \downarrow & = & \downarrow \\ S^1 \setminus \{\theta\} & & [0,1] \end{array}$$

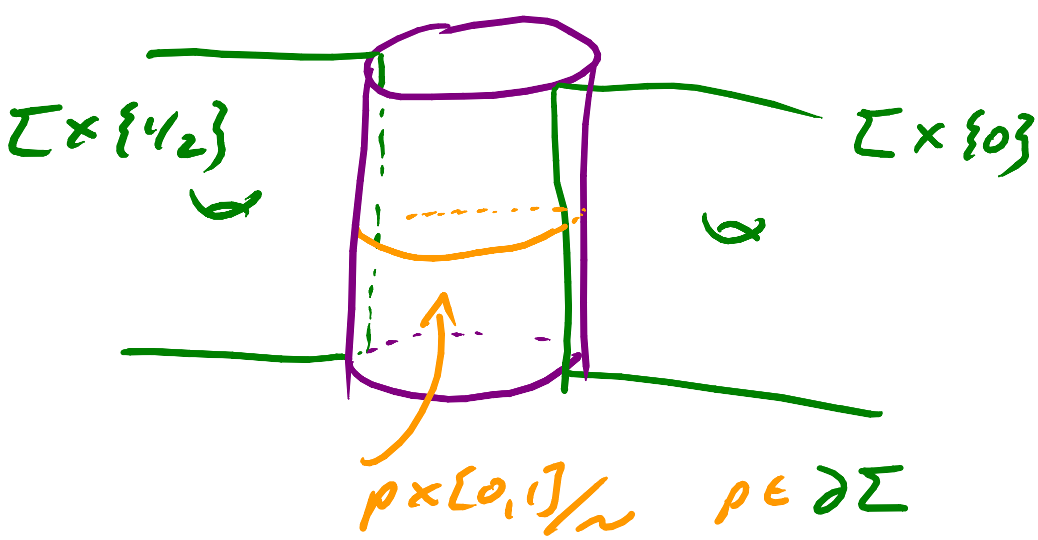
so we recover $M \setminus L$ by gluing $\Sigma \times \{0\}$ to $\Sigma \times \{1\}$ by some diffeomorphism $\phi : \Sigma \to \Sigma$ (that is the identity on $\partial \Sigma$)



write $V_\phi = \Sigma \times [0,1] / (0,\phi(x)) \sim (1,x)$

mapping torus of ϕ

note: for each component of $\partial\Sigma$
V_ϕ has a torus boundary component



we recover M from V_ϕ by collapsing
the circles $p \times [0,1]/\sim$ to points

$$M \cong V_\phi / \{ p \times [0,1]/\sim \}$$

note: Given any surface Σ and diffeo. $\phi: \Sigma \to \Sigma$ ($=id$ on $\partial\Sigma$) we get a manifold

$$M_\phi = V_\phi / \{p \times [0,1]/\sim\}$$

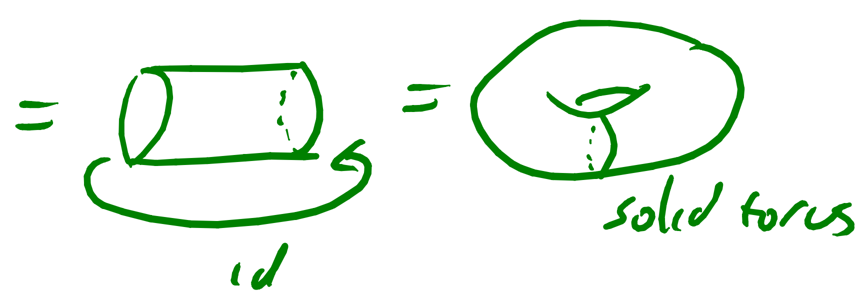
with $L = \{p \times [0,1]/\sim\} \subseteq M_\phi$ binding of o.b.

So we see that we could have defined open books of M to be a pair (Σ, ϕ) (and an identification of M with M_ϕ)

ϕ is called the monodromy of the open book

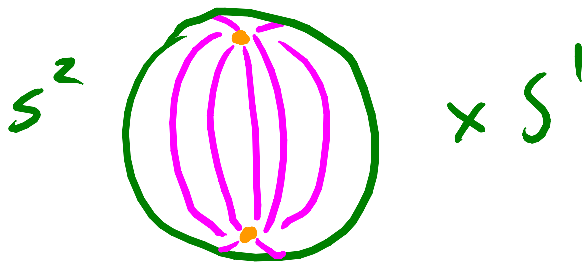
example:

$(D^2, \phi = id)$ gives S^3 with open book given by U ← unknot

$V_\phi =$ 

exercisé!

1) $(S^1\times[0,1], id)$ gives an open book for $S^2\times S^1$



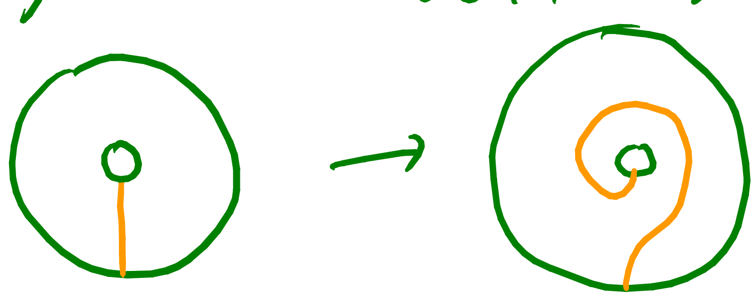
2) Σ a surface of genus g with n boundary components

$\phi = id_\Sigma$

show $M_\phi = \#_{2g+n-1} S^2\times S^1$

3) $\Sigma = S^1\times[0,1]$

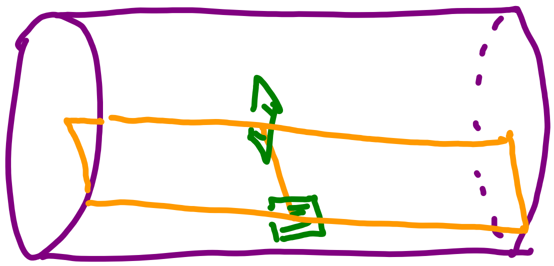
ϕ = right handed Dehn twist



show $M_\phi = S^3$ and binding of the open book is Hopf link

Idea of Thurston-Winkelnkemper:

On $M - V_\phi$ = solid tori use



$\ker(d\phi + r^2 d\theta)$

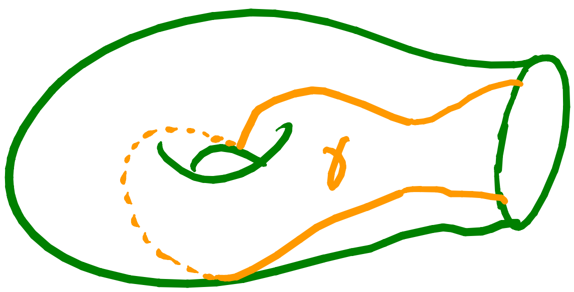
on V_ϕ perturb the tangents to the pages. $\boxed{\text{[1]}}$

Note: The theorem (plus uniqueness remark) gives a well-defined function

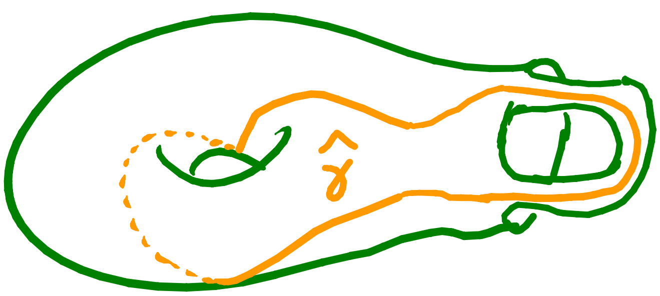
$$\boxed{\Phi : \left\{\begin{matrix}\text{open books}\\ \text{on } M\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{contact}\\ \text{structures}\\ \text{on } M\end{matrix}\right\}}$$

To prove his correspondence Giroux showed Φ is onto and "kernel" is generated by positive stabilization

Given an open book (Σ, ϕ) and any arc γ properly embedded in Σ



let $\Sigma' = \Sigma \cup$ 1-handle
(attached along $\partial\gamma$)



and $\hat{\gamma} = \gamma \cup$ core of 1-handle

set $\phi' = D_{\hat{\gamma}} \circ \phi$

right handed Dehn twist about $\hat{\gamma}$

We say (Σ', ϕ') is a positive stabilization of (Σ, ϕ)

exercise:

1) Show $M_\phi = M_{\phi'}$

2) Determine how the binding changes. (hint: Hopf plumbing)

3) Show supported contact strs are the same

Natural Questions brought up by the Giroux Correspondence

1) Are properties of a contact structure reflected in the open book? Maybe in properties of the page Σ or the monodromy ϕ

2) How does an open book change if we perform some operation on the contact str?

3) How does the contact structure change if we perform some operation on the open book?

The Page and Properties of Contact Strs

Given a contact manifold (M,ξ) let

$$sg(\xi) = \min\{\text{genus}(\Sigma) \mid (\Sigma,\phi) \text{ supports } \xi\}$$

support genus of ξ

Thm (E): If ξ is overtwisted, then $sg(\xi)=0$.

So $sg(\xi)>0$ implies ξ tight!

We have examples of ξ with $sg(\xi)=1$

this comes from an obstruction to $sg=0$

Thm (E):

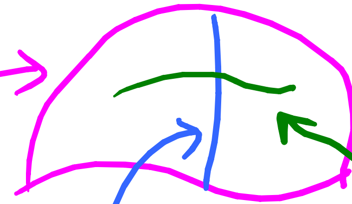
$$\left.\begin{array}{l} sg(\xi)=0 \\ (M,\xi)=\partial(\underset{\text{cpt}}{X},\omega) \end{array}\right\} \Rightarrow b_1(X)=b_2^+(X)=0$$

example: Legedrain surgery on  yields ξ with $sg(\xi)=1$

"Proof":

given

can find a cap such that



D^2 completets to S^2 in $X \cup Y$

S^2 with $S^2 \cdot S^2 = 0$

McDuff $\Rightarrow$ $X \cup Y$ is a blow up of $S^2 \times S^2$ $\blacksquare$

Open Problem: Prove there exists a contact str with $sg > 1$.

Now define

$$bn(\xi) = \min\{\#(\partial\Sigma) \mid (\Sigma,\phi) \text{ supports } \xi \text{ and } g(\Sigma) = sg\}$$

Open Problem: What does $bn(\xi)$ say about ξ?

Guess: There is an upper bound on the Giroux torsion in terms of $bn(\xi)$ (if ξ is tight).

The Monodromy and Properties of ξ

Thm (Loi-Piergallini, Giroux, Akbulut-Ozbagci)

If (M, ξ) is supported by (Σ, ϕ) and ϕ is a composition of positive Dehn twists, then (M, ξ) is Stein fillable.

For some time it was thought

$\xi_{(\Sigma, \phi)}$ Stein fillable

$\Longleftrightarrow$

ϕ composition of pos Dehn twists

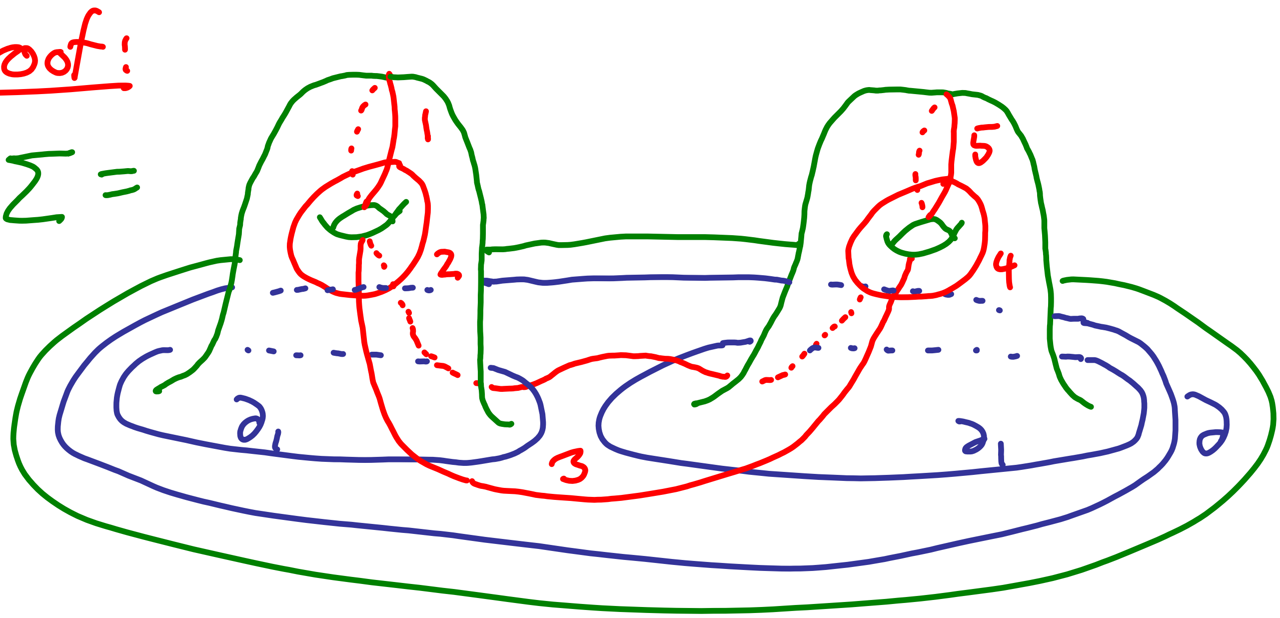
But this is not true!

Thm (Baker-E-Van Horn-Morris, Wand)

There are open books (Σ, ϕ) supporting Stein fillable contact structures for which ϕ cannot be written as a composition of positive Dehn twists

Proof:

$\Sigma =$



$$\Delta = \left(\tau_5\tau_4\tau_3\tau_2\tau_1\tau_5\tau_4\tau_3\tau_2\tau_5\tau_4\tau_3\tau_5\tau_4\tau_5\right)$$

$$\phi = \Delta \circ \tau_{\partial_1}^{-1}\tau_{\partial_2}^{-1}\tau_5\tau_4$$

($\tau_\gamma =$ Dehn twist about γ)

Claim: (Σ, ϕ) supports the tight contact structure ξ on S^3 (which is Stein fillable)

The binding is

(2,1)-cable of (2,3)-torus knot

Suppose ϕ = composition of pos. Dehn twists on n non-separating curves

Can construct a Stein filling of (S^3, ξ) with

1 0-handle
4 1-handles $\Big\}$ comes from $\Sigma \times D^2$
n 2-handles comes from Dehn twists

Eliashberg/Gromov: Any Stein filling of (S^3, ξ) is homeo to B^4

So $\underline{n=4}$ i.e.

$$\phi = \tau_{\gamma_1}\tau_{\gamma_2}\tau_{\gamma_3}\tau_{\gamma_4}$$
$$\parallel$$
$$\Delta\, \tau_{\partial_1}^{-1}\tau_{\partial_2}^{-1}\tau_5\tau_4$$

Can easily check that $\Delta^2 = \tau_\partial$

so
$$\tau_\partial = \Big(\tau_{\gamma_1}\tau_{\gamma_2}\tau_{\gamma_3}\tau_{\gamma_4}\tau_4^{-1}\tau_5^{-1}\overbrace{(\tau_5\tau_4)^6}^{\tau_{\partial_2}}\overbrace{(\tau_1\tau_2)^6}^{\tau_{\partial_1}}\Big)^2$$

= composition of 26×2 pos twists about non-sep. curves

Fact: $H_1(\mathrm{Map}(\Sigma_2)) = \mathbb{Z}/10\mathbb{Z}$
and any Dehn twist about
any non-sep curve represents
a given generator

If we cap off Σ to get Σ_2 we
see $\tau_\partial = id$ so 10 must divide
52

$\therefore$ ϕ has no positive factorization

Recall a **monoid** is a set G with
a multiplication that is associative
and has identity
(that is a "group without inverses")

Given a surface Σ with boundary let $\mathrm{Map}^+(\Sigma)$ be the mapping class group of orientation pres. diffeos of Σ that are the identity on $\partial\Sigma$

We know from above that

$$\phi \in \mathrm{Map}^+(\Sigma) \rightsquigarrow \begin{cases} M_\phi \\ \xi_\phi \end{cases}$$

3-manifold

contact structure

Given a property P of contact structures, let

$$\mathrm{Map}_P(\Sigma) = \{\phi \in \mathrm{Map}^+(\Sigma) : \xi_\phi \text{ has property } P\}$$

Is $\mathrm{Map}_P(\Sigma)$ a monoid?

sometimes yes, sometimes no

examples:

1) $\mathcal{P}$ = tight denote $\mathrm{Map}_{\mathcal{P}}(\Sigma)$ by $\mathrm{Tight}(\Sigma)$

2) $\mathcal{P}$ = Stein fillable denote $\mathrm{Map}_{\mathcal{P}}(\Sigma)$ by $\mathrm{Stein}(\Sigma)$

[$\exists$ complex 4-mfd. (X,J) that properly embedds in $\mathbb{C}^N$ s.t. "$M = \partial X$"
$\xi = T\partial X \cap J\,T\partial X$]

3) $\mathcal{P}$ = universally tight denote $\mathrm{Map}_{\mathcal{P}}(\Sigma)$ by $UT(\Sigma)$

[ξ pulled back to the universal cover of M is tight]

4) $\mathcal{P}$ = strongly fillable $\iff$ [$\exists$ a symplectic mfd. (X, ω) and a vector field $v \pitchfork \partial X = M$ s.t. $\mathcal{L}_v \omega = \omega$, $\xi = \ker(\iota_v \omega)|_{\partial M}$]

denote $Map_{\mathcal{P}}(\Sigma)$ by $Strong(\Sigma)$

5) $\mathcal{P}$ = weakly fillable $\iff$ [$\exists$ a symplectic mfd (X, ω) such that $\omega|_{\xi}$ non-degenerate]

denote $Map_{\mathcal{P}}(\Sigma)$ by $Weak(\Sigma)$

Let $Dehn^+(\Sigma)$ = compositions of positive Dehn twists

It is known that

$$\begin{array}{c} UT \\ \rotatebox{90}{$\subsetneq$} \end{array}$$

$$\text{Dehn}^+ \subsetneq \text{Stein} \subsetneq \text{Strong} \subsetneq \text{Weak} \subsetneq \text{Tight}$$

Thm (Baldwin, Baker–E–Van Horn-Morris 2010):

> Stein, Strong, Weak are monoids
> UT is not a monoid

Major Open Question: Is Tight a monoid? ($\Leftrightarrow$ Legendrian surgery preserves tightness)

Breaking News! Yes

see Andy Wand's talk!

Fact (Wendl): If Σ is planar then

$$\text{Dehn}^+ = \text{Stein} = \text{Strong} = \text{Weak}$$

W + Niederkrüger

So we can characterize these monoids if Σ planar.

Questions

1) Can you characterize when a given ϕ is in one of the above monoids?

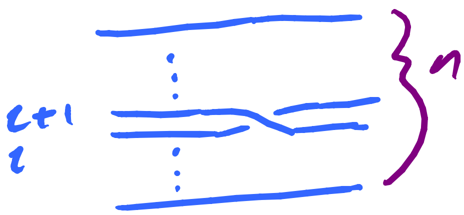
2) If ϕ in one of the monoids then is there a condition to force it into a submonoid?

3) Are these monoids "easily" presented?
   Finitely generated?
   Finitely presented?

4) Are there other monoids in $Map^+(\Sigma)$? Do they correspond to anything interesting in the contact world?

Positivity in the Braid Group

The standard generators of the n-strand braid group $B(n)$ are:



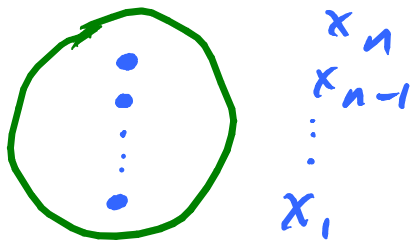
σ_i

so any braid is a word in $\sigma_1 \ldots \sigma_{n-1}, \sigma_1^{-1} \ldots \sigma_{n-1}^{-1}$
and a braid is called <u>positive</u> if it is a word in $\sigma_1 \ldots \sigma_{n-1}$

<u>Note:</u> this notion of positive depends on the generators we chose for $B(n)$
But are the σ_i the most "natural" generators?

Recall that a braid can be thought of as a loop in the configuration space of n points in D^2: $C(D^2, n)$

Indeed, consider loops based at

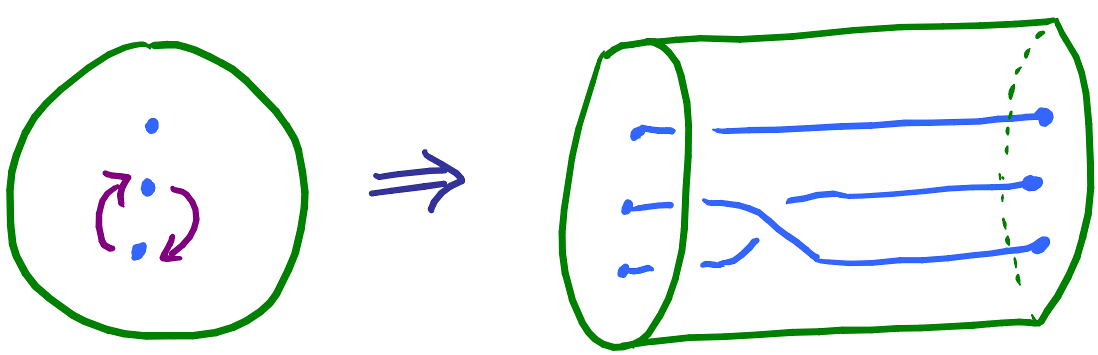


given such a loop we get a braid by thinking of the loop as an isotopy of $\{x_1 \ldots x_n\} \xrightarrow{f_t} D^2$ and looking at the trace of the isotopy

$$\text{image}\{\phi : \{x_1 \ldots x_n\} \times [0,1] \to D^2 \times [0,1]\}$$

$$\phi(x,t) = (f_t(x), t)$$

<u>example:</u>



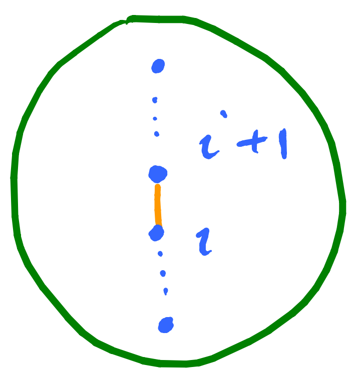
f_t = half twist exchanging x_1, x_2

trace is the braid σ_1

similarly given a braid B think of it as sitting in $D^2 \times [0,1]$

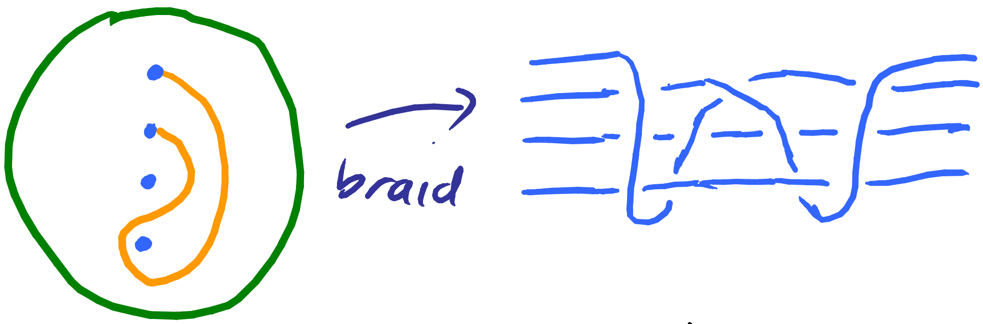
then each $(D^2 \times \{t\}) \cap B$ is an element in the configuration space $C(D^2, n)$

In this language the σ_i are isotopies exchanging the i and $(i+1)$ points by a right handed twist in a nbhd of an arc connecting them



<u>Quasi-Positive Generators:</u> a more "natural" generating set would be the set of right handed twists in a neighborhood of any arc connecting any points!

example: $n=4$



note: this can be written

$$(\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2^{-1})\sigma_3(\sigma_2\sigma_1^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_3^{-1})$$

$$= w\sigma_3 w^{-1}$$

where $w=\sigma_3\sigma_2\sigma_1\sigma_1\sigma_2^{-1}$

- so these generators are just conjugates of the standard generators
- we call a braid quasi-positive if it can be written as a product of conjugates of the σ_i
- you might think this is a more natural notion of positive since it does not rely on chosing special arcs to twist along

- Of course one draw back to this is the quasi-positive generating set is infinite...
- Quasi-positive has another geometic interpretation

Think of S^3 a sphere of some radius in $\mathbb{C}^2$

let Σ be a complex curve in $\mathbb{C}^2$
(for example the zeros of a two variable polynomial)

if $\Sigma \pitchfork S^3$, then $K = \Sigma \cap S^3$
is called a transverse $\mathbb{C}$-link

(note this includes links of singularities, like torus knots, but is a much bigger class of knots)

A link K is called quasi-positive if it is the closure of a quasi-positive braid.

Thm:

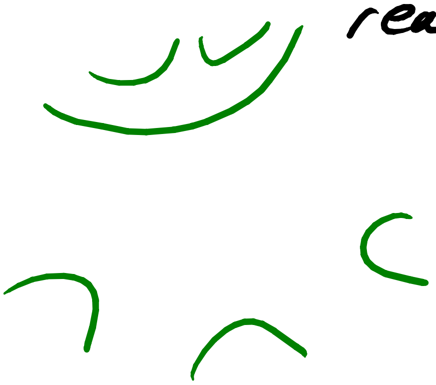
$$\boxed{\{\text{transverse } \mathbb{C}\text{-links}\} = \{\text{quasi-positive links}\}}$$

So quasi-posositivity has a geometric meaning!

Remark: $\supseteq$ by Rudolph 1983

$\subseteq$ by Boileau-Orevkov 2001

Orevkov has a method to use quasi-positivity to study Hilbert's 16^{th} problem: the configurations of real algebraic curves in $\mathbb{R}^2$



ok

not ok ("for M-sextics")

In his study Orevkov asked two questions

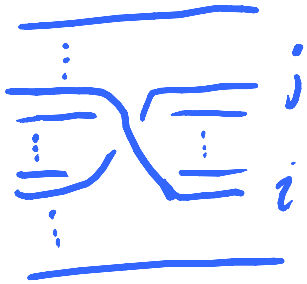
- Given two quasi-positive braids representing a fixed link, are they related by <u>positive</u> Markov moves and conjugation?

- Given a quasi-positive link, is any minimal braid index representative of the link quasi-positive?

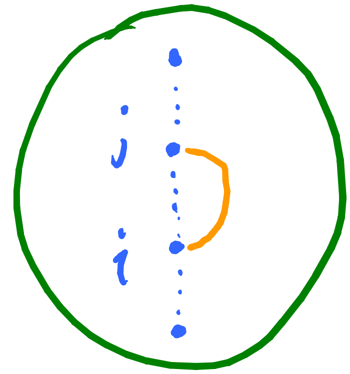
We will give partial answers to these questions using contact geometry.

But first we want to discuss another type of positivity and surfaces that closures of braids bound.

We now define the strongly quasi-positive generators of the braid group $B(n)$: let σ_{ij} be



in terms of the config space



Note:
- there are a finite number of these
- $\sigma_{ij} = (\sigma_i \cdots \sigma_{j-2})\, \sigma_{j-1}\, (\sigma_i \cdots \sigma_{j-2})^{-1}$

  so these are also quasi-positive generators
- a link is strongly quasi-positive if it is the closure of a braid that can be written as a word in the σ_{ij}.

(the surface for K that comes from a strongly qp-presentation is embedded in S^3!)

Note we have the following submonoids in the braid group

$$\left\{\begin{matrix}\text{closure of} \\ \text{positive} \\ \text{braids}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{strongly} \\ \text{qp links}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{qp} \\ \text{links}\end{matrix}\right\} = \left\{\begin{matrix}\text{transv.} \\ \mathbb{C}\text{-links}\end{matrix}\right\}$$

We have good characterizations of these monoids but how do they relate to contact geometry?

Contact Geometry and Positivity

Thm (E-Van Horn-Morris):
let K be 1) fibered
2) strongly quasi-positive
link in S^3
and Σ is associated Seifert surface
Then there is a unique transverse knot in (S^3, ξ_{std}) in the knot type of K with $sl = -\chi(\Sigma)$.

the proof of this is quite involved but we discuss some corollories.

Cor 1 (a recognition result):
let K be a fibered, strongly qp link and B a braid representing K
Then
B is quasi-pos. $\Leftrightarrow$ $a(B) = n(B) - \chi(K)$

alg length of B braid index

Cor 2: Let K be a strongly quasi-pos. fibered knot
Any two quasi-positive braids are related by positive Markov moves (and conjugation)

Note: 1) In particular, two positive braids represent the same knot $\Leftrightarrow$ they are related by positive Markov moves.

2) So all questions about knots represented by positive braids can be answered in the Positive Braid Monoid

3) Answer to Orevkov is
Yes for fibered strongly q.p.
No in general due to examples of Birman and Menasco.

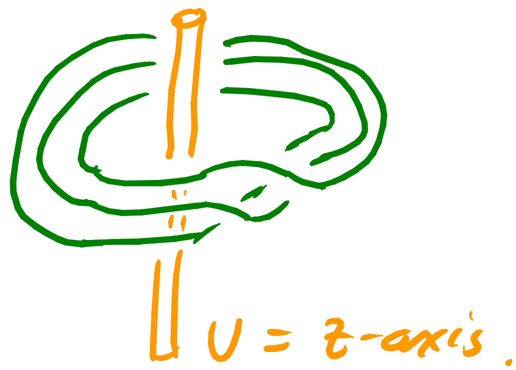
Cor 3:

$\boxed{\text{If } K \text{ is a fibered strongly quasi-pos knot then any minimal braid index representative of } K \text{ is quasi-positive}}$

(this corollary as stated relies on the recent solution to the generalized Jones / Kawamura geography conjecture by Dynnikov and Prasolov and LaFountain and Menasco)

Generalized Braids:

Recall, we say a knot K in S^3 <u>braided</u> if it is transverse to the pages of the open book (U, π)



In general, given any open book (L, π) of M we say K is <u>braided about L</u> if $K \cap L = \emptyset$ and K is transverse to the pages of (L, π).

We have the following generalization of Alexander + Markov

Th$^{\underline{m}}$ (Skora 1992, Sundheim 1993)

- any knot K in M can be braided about any open book (L, π) of M.
- two "braids" are isotopic as knots

$\iff$

they are related by "braid isotopy" and Markov moves

We also have

Benniquin for (S^3, ξ_{std})

Th$^{\underline{m}}$ (Pavelescu 2008):

let ξ be the contact structure supported by the open book (L, π) for M

- any transverse knot K can be braided about (L, π)
- two braids are isotopic through transverse knots

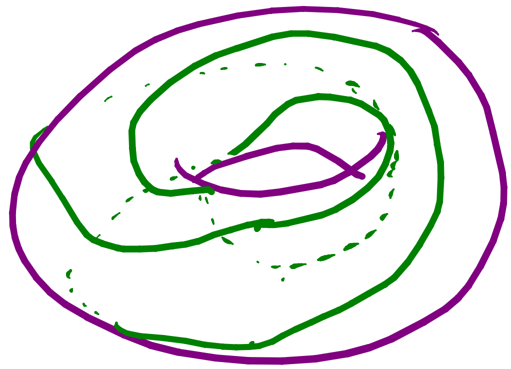
$\iff$

they are related by braid isotopy and positive Markov moves

These theorems prompt a huge number of interesting questions.

For example:

1) on S^3 all the positive torus knots $K_{p,q}$



$K_{3,2}$

are fibered.

So given any K we can braid it about any $K_{p,q}$ and it has a <u>(p,q)-braid index</u> $b_{p,q}(K)$

<u>exercise:</u> $b_{p,q}(K) = 1$ for some p,q.

<u>question:</u> how good of a knot invariant is the set of all $b_{p,q}(K)$?

Since all the $K_{p,q}$ $(p,q>0)$ support the standard contact str on S^3 we can ask the same question for transverse knots

2) Given any (L,π) for M and a sequence of + stabilizations

$$(L,\pi) \xrightarrow[+\text{ stab}]{} (L_1,\pi_1) \xrightarrow[+\text{ stab}]{} (L_2,\pi_2) \to \cdots$$

we again get an infinite sequence of braid indicies b_k for a knot

<u>exercise</u>: $b_k(K) \geq b_{k+1}(K)$

<u>question</u>: Is $b_k(K)$ alway 1 for some k?

How good of an invt are these?

3) Understand the algebraic structure of the generalized braid groups, and how they relate to knots.

4) Define and characterize monoids in the generalized braid groups (e.g. What is (strong) quasi-positivity and is it related to the geometry of the associated contact structure or fillings of it?)

5) Understand how braid stabilization and open book stabilization interact.

Thanks

for

Your

Attention!